Math 3070/6070 Introduction to Probability

Mon/Wed/Fri 9:00am - 9:50am

Instructor: Dr. Xiang Ji, xji4@tulane.edu

Lecture 1:Aug 19

Today

- Introduction
- Introduce yourself
- Course logistics

What is this course about?

This course will provide a calculus-based introduction to probability theory. Material covered will include fundamental axioms of probability, combinatorics, discrete and continuous random variables, multivariate distributions, expectation, and limit theorems, generally following Chapters 1-5 of the textbook. This course is a critical prerequisite for more advanced work in statistical theory and analysis.

Prerequisite

- Calculus

Why learn probability

- The subject of probability theory is the foundation upon which all of statistics is built.
- It provides you a tool to model
  - populations
  - experiments
  - almost anything else that could be considered a random phenomenon
  - example topics in Data Analysis course
- Through these models, statisticians are able to draw inferences about populations based on examination of only a part of the whole.
- A must have for any Data Scientists.

What this course WILL NOT do for you

It will not help you:

- Beat the casino at blackjack (although it may convince you that it is better not to gamble, or that a casino is a great business).
- Answer your friends' silly questions such as "What are the chances it will rain tomorrow?" (although it might make you think of ways that you might model and compute it).

Syllabus

Check course website frequently for updates and announcements.

https://tulane-math-3070-2024.github.io/

HW submission

Students are required to submit hand-written homework in recitations to the TA. Homework assignments are expected every two weeks with 4-5 problems at a time.

Year 2023 comments

Your experience in this course:

- Professor Ji is awesome
- /
- There were a few times where I felt like Professor Ji's comments were a bit condescending. There were times that he would expect more from us to show proofs etc. and he would give us more time because he would assume that we were not capable enough to finish these proofs. He would then chuckle to himself, which felt condescending and really unmotivated me to go to more classes. I think there were a few instances where we also called out specific students in the class, which definitely humiliated them, but also made me uncomfortable.
- Professor Ji is a good lecturer, in my opinion. I heard many complaints that he just reads off of a pdf, but I feel his pdf is well-organized, meaning that the lecture is both easy to understand and easy to review.
- Zoom option is nice.
- Classes consist of him reading out of a textbook that is already difficult to understand just by reading. Very little is added to help you learn the material.
- It was okay at best. I did just as well on the midterm by joining the Zoom in contrast to going to class as I left the class confused 99% of the time. I taught the material to myself because Professor Ji teaches on a pdf. Managed to get a good grade because I

was decent at memorizing and wrote the right things on my test cheat sheet. Ask me in a month, and I will be unable to recall anything from this class.

- Professor Ji, despite your best efforts to scare everyone away from taking this class you clearly do want us to succeeded, which I appriciate. However you could really improve the lecture time. I think that if you actually want people to come to class, you should not allow zoom. I think making attendance mandatory did not work, it just encouraged people to log into zoom. I think that if you would like people to engage more you should find a way to teach that is better than projecting a pdf. I understand that you loath writing, but reading a projected pdf is an incredibly dull way to be presented information. Even slides would be better. Writing would absolutely be the best. I think that if you want people to actually try the examples in class, you should provide clearer problems. The way you do it now is often confusing. You will show us about half of a proof or a theorem that isnt just an inequality and then say "finish this". On more than one occasion, I have found myself spending most of the two-three minutes you gave us to solve the problem just trying to figure out what you were asking. Finally, I think that you could foster a better learning enviroment. By this I mean activly encouraging questions and not laughing at your students when they ask you questions that are obvious to you. I know you are not doing this out of judgment but it does not feel great to be laughed for not understanding what you meant when you explained something the first time. Beyond this, I am a student who will ask questions no matter how to stupid I look because I think there is no other way to learn. However, I bet many other students would ask more questions if you responded in a more encouraging way.
- nice professor
- Teaching from pre-written notes rather than writing things out on the board is fine, especially since he said that was his style at the beginning of the semester. My only gripe would be that the tests were more about reciting definitions rather than applying what we learned to different problems.
- The topics in this course are pretty advanced, and I didn't understand the motivation behind a lot of it. However, the course really helped me understand where commonly used probability models come from, and that was interesting. I also really appreciated the zoom option that Dr. Ji provided for lecture, and he really seems to care about his performance as a professor. He often asked for feedback from students, and provided us the opportunity to vote on syllabus changes and class policies. He also wants his students to do well. The tests were reasonable given the course content and structure, and Dr. Ji is a lenient grader. That being said, I generally did not enjoy this course. There was so much material with extremely involved derivations and proofs that seemed unnecessarily difficult. Lectures were quite boring, and the homework assignments were really long and difficult. Sometimes it felt that Dr. Ji thought the material in this course was simple or easy, which it most definitely was not. That can be pretty discouraging for students who are feeling lost or stuck. Generally speaking though, this course was a good option to fulfill the elective requirement for my math minor and I am glad that I took it.

strongest aspects of this course

- /
- I really liked the way that the exams were organized because they were very straightforward and directly linked to what we had previously learned through the homework, classwork, etc. I also really appreciated Professor Ji's understanding to bring a one page cheatsheet, because I personally feel like memorizing a bunch of formulas is not going to help someone internalize the learning and is futile in the long-run. I think the concept of applying what you learned was a key component throughout this course, and I really liked that.
- I thought that the homework was very helpful, and I actually really liked the lecture style - I don't need the lecturer to write out each theorem or definition; it's definitely enough that Dr. Ji writes out the details of proofs and works out examples on the board.
- Exams very representative
- He went through the proofs in class. Didn't understand them but at least we were exposed to them. He also was super up-to-date with posting and updating class notes which was very convenient.
- I think the way that I truly learned the content of this class was sitting down by myself in the library with the lecture note. They did provide a comprehensive list of everything that we covered. Additionally, the exams reflected what we learned in class and while they were difficult, professor Ji was incredibly forgiving while grading. Thank you for being a kind grader.
- This is the best course I have ever taken. Textbook is awesome. Prof. Ji illustrate examples on the whiteboard very clearly. In particular, I enjoy it on zoom very much!
- He gives a clear roadmap of how to perform well in the class.
- The tests, especially the take home final, and the virtual option for lecture were the strongest aspects of this course.

RateMyProfessor

- (5.0 Quality / 3.0 Difficulty) Very nice and caring professor. Course material is not easy but he grades tests generously.
- (5.0 Quality / 3.0 Difficulty) Great professor. Very easy grader. Thinks he's really funny (he is funny, but his humor isn't for everyone). I learned a little in this class, but it was a manageable course.

Year 2022 comments

Your experience in this course:

- Exactly what I was looking for in a probability course - I got a really good grasp on the theory and this math has already come in useful in other areas and fields that I'm studying. Glad I took this class, and I appreciate the tests being more accessible and spaced out to provide less pressure - highly recommend.
- I really appreciate the lecture shift that Prof. Xiang Ji had after our midterm survey. He took suggestions seriously and dramatically improved how the content was presented to our class's needs. Having the opportunity to present and listen to classmates on related statistical topics was also fun and rewarding.
- Absolutely stupendous course. Fabulous structure, even more fabulous professor.
- Lectures were pretty disengaging. I would've rather had lecture notes written out to us rather than being read to us. Presentation extra credit opportunities were nice. Would've liked more communication and collaboration between TA and teacher.
- Once the lecture structure changed after midterms, I felt I learned much more during lectures. I think that the concepts and theories were explained clearly in class. I appreciate the generosity Professor Ji showed when grading exams, but I think that receiving more detailed feedback would have helped me learn the material better. The exams felt more like a test of our ability to make a formula sheet than a test of our understanding of course material. While this was nice in terms of my grade, I don't think this helped me with my understanding of the material. The recitations often felt disconnected from the course material because we did practice with numerical applications and calculations instead of theory. Overall I feel prepared for the second half of this course next semester.
- I felt the course provided me with very little context for why we were learning about the things we were learning. For example, now I know about a lot of different distributions and their moment generating functions, but I have no idea when I might need to use a Gamma distribution or Poisson. Additionally, I did not find the textbook to be a particularly helpful resource. There also seemed to be little communication between the professor and TA, so the activities we did in recitation weren't always relevant to what we were doing in the class.
- This course and professor were great. I got introduced to an entirely new facet of mathematics and it excited me for my major. Professor X was great and fostered a great learning environment in the classroom.
- I appreciate the professor's willingness to adjust his teaching style to include more written-out derivations. I would have preferred not to have the 10-minute presentations in class on Fridays. I felt like especially at the beginning of the semester most of the content in the presentations was more complex than what we had learned and took class time away from the material. I think it would have been helpful for the TA and Professor to communicate about the level of instruction of the course. It was hard to

understand where I stood from a knowledge perspective when the level of difficulty ranged so greatly from homework, quizzes, worksheets, and lectures. I understand that this class is a probability theory class, however, I think it would be helpful for math majors interested in applied math to have some way to learn more about the applications of probability since the Stats for scientists does not count toward the major.

- Wonderful experience! I learned a tremendous amount, and always left class wanting to know more. This course did not shy away from difficulty and I could not have appreciated it more. Far too often professors dumb down the material in order to cater to the students. Not this class, we learned intense probability theory and I could not be happier with it. I look forward to continuing my studies next semester with Statistical Inference. The rigor and complexity of the course demanded respect and, if given, the knowledge learned is powerful.
- Professor was good, but I think the subject matter was oftentimes too confusing
- The only reasons why I personally did not like the course is firstly because I do not like the subject matter and secondly, I do not like how the material was organized. I do not learn math the best when it is purely based off lecture notes. I did notice that the professor was trying to write more on the board during lectures which I did appreciate.
- I appreciate that Professor Ji mid-semester began to workout problems in class on the whiteboard as that kept me more engaged. Sometimes it was hard to follow the work on the board however as the steps didn't seem to be organized (they looked like they were written all over the place). I think it would be more easy to follow if the notes on the whiteboard were more organized linearly. Also, we only did this like once, but I think it could be a good idea to also give students problems and have them come up to the whiteboard and solve them (maybe for bonus points, doesn't have to be) as that is another way to make class more interactive. I also liked having a practice test as it always nice to get more practice.
- Class sessions were a little slow paced and repetitive for me. However, towards the end as the Professor took some feedback from students and started writing out equations on the board it was easier to follow along and I was more actively learning. I think the weekly presentations were nice but took away a chunk of class time that could've prevented us from getting behind. The weekly reviews were really helpful and I think those are great for Fridays. Our Professor, Dr. Ji, was super accommodating and very adaptive towards creating the best learning environment for students. I appreciated the mid-semester surveys and his changes based off that immensely, and his efforts to supplement our grades with bonus presentations (I just think they could be shorter or maybe uploaded on a discussion post, rather than take up 1 of every 3 classes). I liked that we had a class notes document, rather than a textbook, and this also made the class very accessible when I couldn't be there in person.
- Teaching improved significantly through the semester, he was open to feedback so that helped some. Lectures were still almost entirely him reading off of a pdf though, which did not teach me much

strongest aspects of this course

- Very good in-depth class, useful theory knowledge and strong lectures.
- The strongest aspect of this course is how it provides a very good background which is needed for future statistics courses.
- Loved the professor, absolutely no complaints. Promote Xiang Ji
- Lecture notes helpful. Loved the TA and our lab sessions.
- I really appreciated that Professor Ji asked for student feedback in the middle of the semester and adjusted the lectures based on the results. Once we started doing more derivations on the board, I was able to understand them better.
- I love you as a person, but it is seriously hard to digest anything you say in class. Reading directly from the textbook is not teaching, it is just reading. The tests do not test us on our knowledge of the material AT ALL. They simply did you write down the right thing on your cheat sheet.
- The professor and TA were very flexible when it was clear that the class didn't understand something, and were always open for feedback.
- I really enjoyed having the lecture notes typed and uploaded ahead of class. This allowed me to add to these notes and not have to get a notetaker. I also enjoyed having a notes sheet for the exams especially since there are so many formulas and distributions.
- The rigor and complexity of the course is the strongest aspect. Both Professor Zhao and Professor Ji made the material approachable and were always happy to explain and explain again the difficult concepts and proofs. I am excited to take Statistical Inference for the following semester. Additionally, the course has changed my way of thinking when assessing probabilities in all aspects of life, and there have been many instances over the semester when the knowledge imparted to me has been of service.
- The professor made himself available which I noticed and appreciated. I liked him as a person a lot and I noticed his deep knowledge on the topic.
- Switching to writing on the whiteboard vs pure lecture from typed notes. Enjoyed our bonus presentations on various math topics as it opened my eyes to how versatile statistics can be.
- I listed those above. But the professors attitude and flexibility, and use of technology to sum it up.
- professor was open to student feedback which was helpful
- I liked the presentations a lot. They were a good change of pace and way to understand how this is all applied

RateMyProfessor

- (4.0 Quality / 2.0 Difficulty) Dr Ji has a dry wit and is receptive to student feedback. He is a generous grader and offered an opportunity for generous extra credit. For the tests, he allowed a cheat sheet, and the final was take-home. I also think the tests were easy compared to how complicated they could have been. Beware the class is super theory-based similar to analysis.
- (4.0 Quality / 3.0 Difficulty) At the start of the semester I struggled with Dr. Ji's lecturing style, but after the midterm he asked for our feedback and made adjustments to his class so it was easier to follow his lessons. He didn't always explain things in great detail the first time, but if you ask questions he is always willing to clarify. Exams are also graded generously.
- (3.0 Quality / 3.0 Difficulty) Lectures are based off a pdf document which helps when you need to study, but can be terribly difficult to pay attention to in class. Tests account for about 65% of your grade but he goes pretty easy on the grading.
- (2.0 Quality / 2.0 Difficulty) Don't take this class if you're actually trying to learn the class content. I've been in here a whole semester and genuinely cannot tell you one thing I have retained. Does not communicate with the TA so recitation is not helpful either. Although, homework is graded for completion and quizzes are easy so at least its not a hard grade.
- (4.0 Quality / 5.0 Difficulty) Prof. Xi is hardcore. 3070 is definitely a theory-heavy class for people who really want to get into the underlying technical parts of probability, but if you go into it with that mindset it's really well structured and informative. Book is useful, but you have to be serious and commit time/effort into this class to do well.

Lecture 2: Aug 21

Last time

- Introduction
- Introduce yourself
- Course logistics

Today

- Set theory (1.1)
- Axiomatic Foundations (1.2)
- Calculus of Probabilities (1.2)
- Conditional Probability (1.3)

Set Theory

One of the main objectives of a statistician is to draw conclusions about a population of objects by conducting an experiment. The first step in this endeavor is to identify the possible outcomes or, in statistical terminology, the *sample space*.

Definition The set, S, of all possible outcomes of a particular experiment is called the *sample space* for the experiment.

Example The sample space of

- tossing a coin just once, contains two outcomes, heads and tails

$$S = \{H, T\}$$

- observing reported SAT scores of randomly selected students at a certain university

$$S = \{200, 210, 220, \ldots, 780, 790, 800\}$$

- an experiment where the observation is reaction time to a certain stimulus

$$S = (0, \infty)$$

Definition An *event* is any collection of possible outcomes of an experiment, that is , any subset of S (including S itself).

Let A be an event,

- A is a subset of S,

- event A occurs if the outcome of the experiment is in the set A,
- we generally speak of the probability of an event, rather than a set.

Set operations:

- Containment:
$$A \subset B \iff x \in A \implies x \in B$$
- Equality:
$$A = B \iff A \subset B \text{ and } B \subset A$$
- Union: the union of A and B, written as $A \cup B$, is the set of elements that belong to either A or B or both
$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$
- Intersection: the intersection of A and B, written $A \cap B$, is the set of elements that belong to both A and B:
$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$
- Complementation: the complement of A, written A^c, is the set of all elements that are not in A:
$$A^c = \{x : x \notin A\}.$$

Lecture 3: Aug 23

Last time

- Set theory (1.1)

Today

- Set theory (1.1)
- Axiomatic Foundations (1.2)

Theorem For any three events, A, B, and C, defined on a sample space S,

1. Commutativity
$$A \cup B = B \cup A,$$
$$A \cap B = B \cap A;$$

2. Associativity
$$A \cup (B \cup C) = (A \cup B) \cup C,$$
$$A \cap (B \cap C) = (A \cap B) \cap C;$$

3. Distributive Laws
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C),$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C);$$

4. DeMorgan's Laws
$$(A \cup B)^c = A^c \cap B^c,$$
$$(A \cap B)^c = A^c \cup B^c;$$

We show the proof of $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ in the distributive laws. Caution: Venn diagrams are helpful in visualization, but they do not constitute a formal proof. To prove that two sets are equal, we need to show that each set contains the other.
proof:

- $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$:
Let $x \in (A \cap (B \cup C))$. By definition of intersection, $x \in (B \cup C)$ that is, either $x \in B$ or $x \in C$. Since x also must be in A, we have that either $x \in (A \cap B)$ or $x \in (A \cap C)$; therefore, $x \in ((A \cap B) \cup (A \cap C))$.
- $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$:
Let $x \in ((A \cap B) \cup (A \cap C))$. This implies that $x \in (A \cap B)$ or $x \in (A \cap C)$. If $x \in (A \cap B)$, then x is in both A and B. Since $x \in B$, then $x \in (B \cup C)$and thus $x \in (A \cap (B \cup C))$. It follows the same argument when $x \in (A \cap C)$, we still have $x \in (A \cap (B \cup C))$.

Definition Two events A and B are *disjoint* (or *mutually exclusive*) if $A \cap B = \varnothing$. The events $A_1, A_2, \ldots$ are *pairwise disjoint* (or *mutually exclusive*) if $A_i \cap A_j = \varnothing$ for all $i \neq j$.

Definition If $A_1, A_2, \ldots$ are pairwise disjoint and $\cup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup \cdots = S$, then the collection of $A_1, A_2, \ldots$ forms a *partition* of S.

Example The sets $A_i = [i, i+1), i = 0, 1, 2, \ldots$ form a partition of $[0, \infty)$.

Lecture 4: Aug 26

Last time

- Set theory (1.1)

Today

- Axiomatic Foundations (1.2)

Basics of Probability Theory

When an experiment is performed, the realization of the experiment is an outcome in the sample space. If the experiment is performed a number of times, then

- different outcomes may occur each time
- some outcomes may repeat
- the "frequency of occurrence" of an outcome can be thought of as a probability

However, we **do not** define probabilities in terms of frequencies but instead take the mathematically simpler axiomatic approach. The axiomatic approach is not concerned with the interpretations of probabilities, but is concerned only that the probabilities are defined by a function satisfying the axioms. Interpretations of the probabilities are quite another matter:

- The "frequency of occurrence" of an event is one example of a particular interpretation of probability.
- Another possible interpretation is a subjective one, where we can think of the probability as a belief in the chance of an event occurring.

Axiomatic Foundations

For each event A in the sample space S, we want to associate with A a number between zero and one that will be called the probability of A, denoted by $\Pr(A)$. The domain of Pr is the set where the arguments of the function $\Pr(\cdot)$ are defined. It is natural to define the domain of Pr as all subsets of S, that is for each $A \subset S$, we define $\Pr(A)$ as the probability that A occurs. However, there are some technical difficulties to overcome which requires us to familiarize with the following.

Definition A collection of subsets of S is called a *sigma algebra* (or *Borel field*), denoted by $\mathcal{B}$, if it satisfies the following three properties:

1. $\varnothing \in \mathcal{B}$ (the empty set is an element of $\mathcal{B}$).
2. If $A \in \mathcal{B}$, then $A^c \in \mathcal{B}$ ($\mathcal{B}$ is closed under complementation).
3. If $A_1, A_2, \cdots \in \mathcal{B}$, then $\cup_{i=1}^{\infty} A_i \in \mathcal{B}$ ($\mathcal{B}$ is closed under countable unions).

From Property (1) and (2), we see that the empty set and its complement S (since $S = \varnothing^c$) are always in a sigma algebra. In fact, they construct the *trivial* algebra $\{\varnothing, S\}$ which is the smallest sigma algebra.

By DeMorgan's Law, (3) can be replaced by:

$$3'. \text{ if } A_1, A_2, \cdots \in \mathcal{B}, \text{ then } \cap_{i=1}^{\infty} A_i \in \mathcal{B}.$$

This is because:

$$(\cup_{i=1}^{\infty} A_i^c)^c = \cap_{i=1}^{\infty} A_i.$$

Example If S is finite or countable (where the elements of S can be put into one to one correspondence with a subset of the integers), then these technicalities really do not arise, for we define for a given sample space S,

$$\mathcal{B} = \{\text{all subsets of } S, \text{ including } S \text{ itself}\}.$$

If S has n elements, there are 2^n sets in $\mathcal{B}$ (why?).[hint: for each element, it is either in or out of a subset, so 2 choices].

Example Let $S = (-\infty, \infty)$, be the real line. Then $\mathcal{B}$ is chosen to contain all sets of the form

$$[a, b], (a, b], (a, b), \text{ and } [a, b)$$

for all real numbers a and b. Also, from the properties of $\mathcal{B}$, it follows that $\mathcal{B}$ contains all sets that can be formed by taking (possibly countably infinite) unions and intersections of sets of the above varieties.

We now define a probability function.

Definition Given a sample space S and an associated sigma algebra $\mathcal{B}$, a *probability function* is a function Pr with domain $\mathcal{B}$ that satisfies

1. $\Pr(A) \geqslant 0$ for all $A \in \mathcal{B}$.
2. $\Pr(S) = 1$.
3. If $A_1, A_2, \cdots \in \mathcal{B}$ are pairwise disjoint, then $\Pr(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \Pr(A_i)$.

The above three properties are usually referred to as the Axioms of Probability (or the Kolmogorov Axioms, after A. Kolmogorov, one of the fathers of probability theory). Any function that satisfies the Axioms of Probability is called a probability function.

Example Consider the simple experiment of tossing a fair coin (just once), so $S = \{H, T\}$. A reasonable probability function is the one that assigns equal probabilities to heads and tails, that is,

$$\Pr(\{H\}) = \Pr(\{T\}).$$

Since $S = \{H\} \cup \{T\}$, we have , from Axiom 2, $\Pr(\{H\} \cup \{T\}) = 1$. Also, $\{H\}$ and $\{T\}$ are disjoint, so $\Pr(\{H\} \cup \{T\}) = \Pr(\{H\}) + \Pr(\{T\})$. Collectively, we have

$$\begin{aligned}
&\Pr(\{H\}) = \Pr(\{T\}) \\
&\Pr(\{H\} \cup \{T\}) = 1 \\
&\Pr(\{H\} \cup \{T\}) = \Pr(\{H\}) + \Pr(\{T\})
\end{aligned}$$

Therefore, $\Pr(\{H\}) = \Pr(\{T\}) = \frac{1}{2}$.

Example If S is finite or countable (where the elements of S can be put into $1-1$ correspondence with a subset of the integers), then these technicalities really do not arise, for we we define for a given sample space S,

$$\mathcal{B} = \{\text{all subsets of } S, \text{ including } S \text{ itself}\}.$$

If S has n elements, there are 2^n sets in $\mathcal{B}$ (why?).[hint: for each element, it is either in or out of a subset, so 2 choices].

Example Let $S = (-\infty, \infty)$, the real line. Then $\mathcal{B}$ is chosen to contain all sets of the form

$$[a, b], (a, b], (a, b), \text{ and } [a, b)$$

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Lecture 5: Aug 28

Last time

- Axiomatic Foundations (1.2)

Today

- Axiomatic Foundations (1.2)
- Calculus of Probabilities (1.2)

Example Consider the simple experiment of tossing a fair coin (just once), so $S = \{H, T\}$. A reasonable probability function is the one that assigns equal probabilities to heads and tails, that is,

$$\Pr(\{H\}) = \Pr(\{T\}).$$

Since $S = \{H\} \cup \{T\}$, we have , from Axiom 1, $\Pr(\{H\} \cup \{T\}) = 1$. Also, $\{H\}$ and $\{T\}$ are disjoint, so $\Pr(\{H\} \cup \{T\}) = \Pr(\{H\}) + \Pr(\{T\})$. Collectively, we have

$$\begin{aligned}
&\Pr(\{H\}) = \Pr(\{T\}) \\
&\Pr(\{H\} \cup \{T\}) = 1 \\
&\Pr(\{H\} \cup \{T\}) = \Pr(\{H\}) + \Pr(\{T\})
\end{aligned}$$

Therefore, $\Pr(\{H\}) = \Pr(\{T\}) = \frac{1}{2}$.

Caculus of Probabilities

We start with some fairly self-evident properties of the probability function when applied to a single event.

Theorem If Pr is a probability function and A is any set in $\mathcal{B}$, then

1. $\Pr(\varnothing) = 0$, where $\varnothing$ is the empty set;
2. $\Pr(A) \leqslant 1$;
3. $\Pr(A^c) = 1 - \Pr(A)$.

proof:

- It's easy to prove (3) first. Since
  - $\Pr(A \cup A^c) = \Pr(S) = 1$,
  - A and A^c are disjoint, by axiom (3), $\Pr(A \cup A^c) = \Pr(A) + \Pr(A^c)$.

  so that $\Pr(A) + \Pr(A^c) = \Pr(S) = 1$
- with (3) proved, (1) is simple. because we know that

– $S \cup \varnothing = S$,

– $S \cap \varnothing = \varnothing$, they are disjoint,

so that $\Pr(\varnothing) + \Pr(S) = \Pr(\varnothing \cup S) = \Pr(S)$.

- now for (2), $\Pr(A) = 1 - \Pr(A^c) \leqslant 1$, by axiom (1).

Formula (2) in the above theorem gives a useful inequality for the probability of an intersection (Bonferroni's Inequality):

$$\Pr(A \cap B) \geqslant \Pr(A) + \Pr(B) - 1.$$

Theorem If Pr is a probability function, then

1. $\Pr(A) = \sum_{i=1}^{\infty} \Pr(A \cap C_i)$ for any partition $C_1, C_2, \ldots$;
2. $\Pr(\cup_{i=1}^{\infty} A_i) \leqslant \sum_{i=1}^{\infty} \Pr(A_i)$ for any sets $A_1, A_2, \ldots$.

where (1) is also referred to as "Total probability" and (2) is Boole's inequality.
proof:
By definition, since $C_1, C_2, \ldots$ form a partition, we have $C_i \cap C_j = \varnothing$ for all $i \neq j$, and $S = \cup_{i=1}^{\infty} C_i$. Therefore,

$$A = A \cap S = A \cap (\cup_{i=1}^{\infty} C_i) = \cup_{i=1}^{\infty} (A \cap C_i),$$

where the last equality follows from the Distributive Law. Since $\{A \cap C_i\} \cap \{A \cap C_j\} = \varnothing$ (i.e. $A \cap C_i$ and $A \cap C_j$ are disjoint), we have

$$\Pr(A) = \Pr\left(\cup_{i=1}^{\infty} (A \cap C_i)\right) = \sum_{i=1}^{\infty} \Pr(A \cap C_i).$$

To establish Boole's Inequality, we first construct a disjoint collection $A_1^*, A_2^*, \ldots$, with the property that $\cup_{i=1}^{\infty} A_i^* = \cup_{i=1}^{\infty} A_i$. We define A_i^* by

$$A_1^* = A_1,\ A_i^* = A_i \backslash \left(\cup_{j=1}^{i-1} A_j\right),\ i = 2, 3, \ldots,$$

where the notation $A \backslash B$ denotes the part of A that does not intersect with B. In other words, $A \backslash B = A \cap B^c$. It's easy to see that $\cup_{i=1}^{\infty} A_i^* = \cup_{i=1}^{i} nftyA_i$, and we have

$$\Pr\left(\cup_{i=1}^{\infty} A_i\right) = \Pr\left(\cup_{i=1}^{\infty} A_i^*\right) = \sum_{i=1}^{\infty} \Pr(A_i^*)$$

where the last equality holds because A_i^* are disjoint. To see this, consider any pair of $A_i^* \cap A_k^*, i > k$, then

$$\begin{aligned} A_i^* \cap A_k^* &= \left\{A_i \backslash (\cup_{j=1}^{i-1} A_j)\right\} \cap \left\{A_k \backslash (\cup_{j=1}^{k-1} A_j)\right\} \\ &= \left\{A_i \cap (\cup_{j=1}^{i-1} A_j)^c\right\} \cap \left\{A_k \cap (\cup_{j=1}^{k-1} A_j)^c\right\} \\ &= \left\{A_i \cap \left(\cap_{j=1}^{i-1} A_j^c\right)\right\} \cap \left\{A_k \cap \left(\cap_{j=1}^{k-1} A_j^c\right)\right\} \\ &= \varnothing. \end{aligned}$$

Lastly, we have $\Pr(A_i^*) \leqslant \Pr(A_i)$.

Lecture 6: Aug 30

Last time

- Calculus of Probabilities (1.2)

Today

- no class next Monday (Labor day)
- Binomial theorem
- Conditional Probability (1.3)
- Independence (1.3)

Theorem If Pr is a probability function, then

1. $\Pr(A) = \sum_{i=1}^{\infty} \Pr(A \cap C_i)$ for any partition $C_1, C_2, \ldots$;
2. $\Pr(\cup_{i=1}^{\infty} A_i) \leqslant \sum_{i=1}^{\infty} \Pr(A_i)$ for any sets $A_1, A_2, \ldots$.

where (1) is also referred to as "Total probability" and (2) is Boole's inequality.
proof:
By definition, since $C_1, C_2, \ldots$ form a partition, we have $C_i \cap C_j = \varnothing$ for all $i \neq j$, and $S = \cup_{i=1}^{\infty} C_i$. Therefore,

$$A = A \cap S = A \cap (\cup_{i=1}^{\infty} C_i) = \cup_{i=1}^{\infty} (A \cap C_i),$$

where the last equality follows from the Distributive Law. Since $\{A \cap C_i\} \cap \{A \cap C_j\} = \varnothing$ (i.e. $A \cap C_i$ and $A \cap C_j$ are disjoint), we have

$$\Pr(A) = \Pr\left(\cup_{i=1}^{\infty} (A \cap C_i)\right) = \sum_{i=1}^{\infty} \Pr(A \cap C_i).$$

To establish Boole's Inequality, we first construct a disjoint collection $A_1^*, A_2^*, \ldots$, with the property that $\cup_{i=1}^{\infty} A_i^* = \cup_{i=1}^{\infty} A_i$. We define A_i^* by

$$A_1^* = A_1, \; A_i^* = A_i \backslash \left(\cup_{j=1}^{i-1} A_j\right), \; i = 2, 3, \ldots,$$

where the notation $A \backslash B$ denotes the part of A that does not intersect with B. In other words, $A \backslash B = A \cap B^c$. It's easy to see that $\cup_{i=1}^{\infty} A_i^* = \cup_{i=1}^{\infty} A_i$, and we have

$$\Pr\left(\cup_{i=1}^{\infty} A_i\right) = \Pr\left(\cup_{i=1}^{\infty} A_i^*\right) = \sum_{i=1}^{\infty} \Pr(A_i^*)$$

where the last equality holds because A_i^* are disjoint. To see this, consider any pair of $A_i^* \cap A_k^*, i > k$, then

$$\begin{aligned} A_i^* \cap A_k^* &= \left\{A_i \backslash (\cup_{j=1}^{i-1} A_j)\right\} \cap \left\{A_k \backslash (\cup_{j=1}^{k-1} A_j)\right\} \\ &= \left\{A_i \cap (\cup_{j=1}^{i-1} A_j)^c\right\} \cap \left\{A_k \cap (\cup_{j=1}^{k-1} A_j)^c\right\} \\ &= \left\{A_i \cap \left(\cap_{j=1}^{i-1} A_j^c\right)\right\} \cap \left\{A_k \cap \left(\cap_{j=1}^{k-1} A_j^c\right)\right\} \\ &= \varnothing. \end{aligned}$$

Lastly, we have $\Pr(A_i^*) \leqslant \Pr(A_i)$.

Conditional Probability

All of the probabilities that we have dealt with thus far have been unconditional probabilities. A sample space was defined and all probabilities were calculated with respect to that sample space. In many instances, however, we are in a position to update the sample space based on new information. In such cases we want to be able to update probability calculations or to calculate *conditional probabilities*.

Definition If A and B are events in S, and $\Pr(B) > 0$, then the *conditional probability* of A given B, written $\Pr(A|B)$, is

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}.$$

Note that B becomes the sample space now: $\Pr(B|B) = 1$.

Lecture 7: Sept 4

Last time

- Conditional Probability (1.3)

Today

- Conditional Probability (1.3)
- Independence (1.3)
- Random variables

Example Four cards are dealt from the top of a well-shuffled deck. What is the probability that they are the four aces? What is the probability of getting four aces at the top if knowing the first card is an ace? (there are in total 52 cards)

solution:
We define two events first. Let A be the event $\{4$ aces on top$\}$, and B be the event $\{$the first card on top is an ace$\}$. For a well-shuffled deck, all groups of 4 cards are equally likely. For the 4 aces on top, we have 4! ways of ordering (i.e., permutations of 4 distinct elements where order matters). For the rest of $52-4=48$ cards, there are 48! permutations (where again order matters). Therefore, the probability of event A is $\Pr(A)=\frac{4!48!}{52!}=\frac{4!}{52\cdot 51\cdot 50\cdot 49}=\frac{1}{270{,}725}$.
Note, $\binom{n}{m}$ reads "from n choose m" (for $m\leqslant n$) and calculates by $\binom{n}{m}=\frac{n!}{(n-m)!m!}$ that gives the number of distinct combinations of choosing m elements from n total elements.
Now, let's calculate $\Pr(A|B)$. First of all, $A\subset B$, so that we have $\Pr(A\cap B)=\Pr(A)$. For $\Pr(B)$, having an ace on top instead of the other 12 kinds, $\Pr(B)=\frac{1}{13}$. Then $\Pr(A|B)=\frac{\Pr(A\cap B)}{\Pr(B)}=\frac{\Pr(A)}{\Pr(B)}=\frac{1}{20{,}825}$.

Theorem (Bayes' Rule) Let $A_1, A_2, \ldots$ be a partition of the sample space, and let B be any set. Then, for each $i=1,2,\ldots,$

$$\Pr(A_i|B)=\frac{\Pr(B|A_i)\Pr(A_i)}{\sum_{j=1}^{\infty}\Pr(B|A_j)\Pr(A_j)}.$$

proof:
By "Total probability", we have $\Pr(B)=\sum_{j=1}^{\infty}\Pr(B\cap A_i)$ which is the denominator. Therefore, $\Pr(A_i|B)=\frac{\Pr(A_i\cap B)}{\Pr(B)}=\frac{\Pr(B|A_i)\Pr(A_i)}{\sum_{j=1}^{\infty}\Pr(B\cap A_i)}$.

Lecture 8: Sept 6

Last time

- Conditional Probability (1.3)

Today

- HW2 posted
- Independence
- Random variables

Independence

Definition Two events, A and B, are *statistically independent* if

$$\Pr(A \cap B) = \Pr(A)\Pr(B)$$

Note that independence could have been defined using Bayes' rule by $\Pr(A|B) = \Pr(A)$ or $\Pr(B|A) = \Pr(B)$ as long as $\Pr(A) > 0$ or $\Pr(B) > 0$. More notation, often statisticians omit $\cap$ when writing intersection in a probability function which means $\Pr(AB) = \Pr(A \cap B)$. Sometime, statisticians use comma (,) to replace $\cap$ inside a probability function too, $\Pr(A, B) = \Pr(A \cap B)$.

Theorem If A and B are independent events, then the following pairs are also independent.

1. A and B^c,
2. A^c and B,
3. A^c and B^c.

proof:
For (1),

$$\begin{aligned}\Pr(A, B^c) &= \Pr(A) - \Pr(A, B)\\ &= \Pr(A) - \Pr(A)\Pr(B)\\ &= \Pr(A)(1 - \Pr(B))\\ &= \Pr(A)\Pr(B^c)\end{aligned}$$

For (2), we just need to switch A and B.
For (3), we have A^c and B are independent, then we can treat A^c as A' and B as B', then A' and B'^c are independent which is A^c and B^c are independent.
Alternatively, for (2),

$$\begin{aligned}\Pr(A^c, B) &= \Pr(A^c|B)\Pr(B)\\ &= [1 - \Pr(A|B)]\Pr(B)\\ &= [1 - \Pr(A)]\Pr(B)\\ &= \Pr(A^c)\Pr(B).\end{aligned}$$

And for (3),

$$\begin{aligned}\Pr(A^c, B^c) &= \Pr(A^c) - \Pr(A^c, B) \\ &= \Pr(A^c) - \Pr(A^c)\Pr(B) \\ &= \Pr(A^c)\Pr(B^c).\end{aligned}$$

Example Let the sample space S consist of the 3! permutations of the letters a, b, and c along with the three triples of each letter. Thus,

$$S = \left\{ \begin{array}{ccc} \text{aaa} & \text{bbb} & \text{ccc} \\ \text{abc} & \text{bca} & \text{cba} \\ \text{acb} & \text{bac} & \text{cab} \end{array} \right\}.$$

Furthermore, let each element of S have probability $\frac{1}{9}$. Define

$$A_i = \{i^{th} \text{ place in the triple is occupied by } a\}.$$

What are the values for $\Pr(A_i), i = 1, 2, 3$? Are they pairwise independent?
solution
It is easy to count that

$$\Pr(A_i) = \frac{1}{3}, i = 1, 2, 3,$$

and

$$\Pr(A_1, A_2) = \Pr(A_1, A_3) = \Pr(A_2, A_3) = \frac{1}{9}$$

so that A_is are pairwise independent.

Lecture 9: Sept 9

Last time

- Independence

Today

- Random variables
- Distribution Functions
- Types of Random Variables

Definition* A collection of events $A_1, \ldots, A_n$ are *mutually independent* if for any subcollection $A_{i_1}, \ldots, A_{i_k}$, we have

$$\Pr\left(\cap_{j=1}^{k} A_{i_j}\right) = \prod_{j=1}^{k} \Pr(A_{i_j}).$$

Random Variables

In many experiments, it is easier to deal with a summary variable than with the original probability structure.

Definition A *random variable* (r.v.) is a function from a sample space S into the real numbers.

Example In some experiments random variables are implicitly used

Examples of random variables

Experiment	Random variable
Toss two dice	X = sum of numbers
Toss a coin 25 times	X = number of heads in 25 tosses
Apply different amounts of fertilizer to corn plants	X = yield / acre

In defining a random variable, we have also defined a new sample space (the range of the random variable).

Induced probability function Suppose we have a sample space $S = \{s_1, s_2, \ldots, s_n\}$ with a probability function Pr defined on the original sample space. We define a random variable X with range $\mathcal{X} = \{x_1, \ldots, x_m\}$. We can define a probability function $\Pr_X$ on $\mathcal{X}$ in the

following way. We will observe $X = x_i$ if an only if the outcome of the random experiment is an $s_j \in S$ such that $X(s_j) = x_i$. Therefore,

$$\Pr_X(X = x_i) = \Pr(\{s_j \in S : X(s_j) = x_i\}),$$

defines an *induced* probability function on $\mathcal{X}$, defined in terms of the original function Pr.

We will write $\Pr(X = x_i)$ rather than $\Pr_X(X = x_i)$ for simplicity. Note on notation: random variables will always be denoted with uppercase leeters and the realized values of the variable (or its range) will be denoted by the corresponding lowercase letters.

Example Consider the experiment of tossing a fair coin three times. Define the random variable X to be the number of heads obtained in the three tosses. A complete enumeration of the value of X for each point in the sample space is

s	HHH	HHT	HTH	THH	TTH	THT	HTT	TTT
$X(s)$	3	2	2	2	1	1	1	0

What is the range of X? What is the induced probability function $\Pr_X$?
solution:
The range for the random variable X is $\mathcal{X} = \{0, 1, 2, 3\}$. Assuming all 8 points in S has probability $\frac{1}{8}$. By simply counting, we see that the induced probability function on $\mathcal{X}$ is

x	0	1	2	3
$\Pr_X(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

So far, we have seen finite S and finite $\mathcal{X}$, and the definition of $\Pr_X$ is straightforward. If $\mathcal{X}$ is uncountable, we define the induced probability function, $\Pr_X$ for anyset $A \subset \mathcal{X}$,

$$\Pr_X(X \in A) = \Pr(\{s \in S : X(s) \in A\}).$$

This defines a legitimate probability function for which the Kolmogorov Axioms can be verified.

Distribution Functions

Distribution Functions are used to describe the behavior of a r.v.

Cumulative distribution function

Definition The *cumulative distribution function* or *cdf* of a random variable X, denoted by $F_X(x)$, is defined by

$$F_X(x) = \Pr_X(X \leqslant x), \text{ for all } x.$$

Definition The *survival function* of a random variable X, is defined by

$$S_X(x) = 1 - F_X(x) = \Pr_X(X > x).$$

Example Consider the experiment of tossing three fair coins, and let X = number of heads observed. The cdf of X is

$$F_X(x) = \begin{cases} 0 & \text{if } -\infty < x < 0 \\ \frac{1}{8} & \text{if } 0 \leqslant x < 1 \\ \frac{1}{2} & \text{if } 1 \leqslant x < 2 \\ \frac{7}{8} & \text{if } 2 \leqslant x < 3 \\ 1 & \text{if } 3 \leqslant x < \infty \end{cases}$$

Lecture 10: Sept 11

Last time

- Distribution Functions

Today

- Types of Random Variables
- Discrete Random Variables
- Continuous Random Variables
- Counting Techniques

Some properties of the cdf:
Let $F(x)$ be a cdf. Then

1. $0 \leqslant F(x) \leqslant 1$
2. $\lim_{x \to -\infty} F(x) = 0$
3. $\lim_{x \to \infty} F(x) = 1$
4. F is nondecreasing: if $a < b$, then $F(a) \leqslant F(b)$
5. F is right-continuous: $\lim_{x \downarrow b} F(x) = F(b)$, or $\lim_{x \to b^+} F(x) = F(b)$
6. $\Pr(a < X \leqslant B) = F(b) - F(a)$

Theorem The function $F(x)$ is a cdf if and only if the following three conditions hold:

1. $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$
2. F is nondecreasing: if $a < b$, then $F(a) \leqslant F(b)$
3. F is right-continuous: $\lim_{x \downarrow b} F(x) = F(b)$, or $\lim_{x \to b^+} F(x) = F(b)$

The cdf does not contain information about the original sample space.

Definition Two random variables X and Y are identically distributed if, for every Borel set $A \subset \mathbb{R}$, $\Pr(X \in A) = \Pr(Y \in A)$.

Example Toss a fair coin n times. The number of heads and the number of tails have the same distribution.

Theorem The following two statements are equivalent:

1. The random variables X and Y are *identically distributed*.
2. $F_X(x) = F_Y(x)$ for every x.

Types of Random Variables

Definition A random variable X can be

- *discrete*:
  - X takes on a finite or countably infinite number of values
  - $F_X(x)$ is step-wise constant
- *continuous*:
  - the range of X consists of subsets of the real line
  - $F_X(x)$ is continuous.
- *mixed:* $F_X(x)$ is piecewise continuous.

Example A random variable has cdf

$$F(x) = \begin{cases} 0 & x < 0 \\ x/2 & 0 \leqslant x < 1 \\ 2/3 & 1 \leqslant x < 2 \\ 11/12 & 2 \leqslant x < 3 \\ 1 & 3 \leqslant x \end{cases}$$

Is this a valid cdf? Is it a discrete random variable or continuous random variable or mixed?
solution:
$F(x)$ satisfies the three properties of a cdf that

1. $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$
2. F is nondecreasing: if $a < b$, then $F(a) \leqslant F(b)$
3. F is right-continuous: $\lim_{x \downarrow b} F(x) = F(b)$, or $\lim_{x \to b^+} F(x) = F(b)$.

Therefore, $F(x)$ is a valid cdf. The random variable X is a mixed type.

Lecture 11: Sept 13

Last time

- Types of Random Variables

Today

- Discrete Random Variables
- Continuous Random Variables

Discrete Random Variables

Suppose a random variable X takes only a finite or countable number of values. Let the sample space of X be $S = \{x_1, x_2, \dots\}$. Then the cdf can be expressed as:

$$F(x) = \sum_{x_i \leqslant x} \Pr(X = x_i).$$

Definition The *probability mass function* (pmf) of a discrete random variable X is given by

$$f_X(x) = \Pr(X = x) \text{ for all } x.$$

If the sample space of X is $X = \{x_1, x_2, \dots\}$, then

$$f(x_i) = \Pr(X = x_i) = \Pr(x_{i-1} < X \leqslant x_i) = F(x_i) - F(x_{i-1}).$$

Example (Geometric probabilities) Suppose we do an experiment that consists of tossing a coin until a head appears. Let p = probability of a head on any given toss, and define a random variable X = number of tosses required to get a head. Then for any $x = 1, 2, \dots,$

$$\Pr(X = x) = (1-p)^{x-1}p,$$

since we must get $x - 1$ tails followed by a head for the event to occur and all trials are independent. What is the pmf of the above Geometric distribution? What is the cdf?

solution:
We have the pmf

$$f(x) = \Pr(X = x) = \begin{cases} (1-p)^{x-1}p & \text{for } x = 1, 2, \dots \\ 0 & \text{otherwise.} \end{cases}$$

For cdf, we have

$$\begin{aligned} F(x) = \Pr(X \leqslant x) &= \sum_{i=1}^{\lfloor x \rfloor} f(i) \\ &= \begin{cases} f(1) + f(2) + \cdots + f(\lfloor x \rfloor) & \text{for } x \geqslant 1 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 - (1-p)^{\lfloor x \rfloor} & \text{for } x \geqslant 1 \\ 0 & \text{for } x < 1 \end{cases} \end{aligned}$$

where $\lfloor x \rfloor$ denote the floor function that returns the largest integer smaller or equal to x and we used the summation of a geometric sequence.

Definition The *domain* of a random variable X is the set of all values of x for which $f(x) > 0$. This is also called *range*, *sample space* or *support*.

Properties of the pmf:

1. $f(x) > 0$ for at most a countable number of values x. For all other values x, $f(x) = 0$.
2. Let $\{x_1, x_2, \dots\}$ denote the domain of X. Then

$$\sum_{i=1}^{\infty} f(x_i) = 1.$$

An obvious consequence is that $f(x) \leqslant 1$ over the domain.

Lecture 12: Sept 16

Last time

- Discrete Random Variables

Today

- Continuous Random Variables
- Transformations of Random Variables

Example What is the pmf of a deterministic random variable (a constant)?
solution:

$$f(x) = \Pr(X = x) = \begin{cases} 1 & \text{for } x = c \\ 0 & \text{otherwise.} \end{cases}$$

This is equivalent as a constant of value c.

Example In many applications, a formula can be used to represent the pmf of a random variable. Suppose X can take values $1, 2, \dots$ with pmf

$$f(x) = \begin{cases} \frac{1}{x(x+1)} & \text{for } x = 1, 2, \dots \\ 0 & \text{otherwise.} \end{cases}$$

How would we determine if this is an allowable pmf?
solution:
We show that $f(x)$ satisfies the properties of pmf.

1. $f(x) > 0$ for a countable number of values x. For all other values x, $f(x) = 0$.
2. Let $\{x_1, x_2, \dots\}$ denote the domain of X. Then

$$\sum_{i=1}^{\infty} f(x_i) = \sum_{i=1}^{\infty} f(i) = \sum_{i=1}^{\infty} \left(\frac{1}{i} - \frac{1}{i+1} \right) = 1.$$

Continuous Random Variables

Definition A random variable X is *continuous* if $F_X(x)$ is a continuous function of x.

Definition A random variable X is *absolutely continuous* if $F_X(x)$ is an absolutely continuous function of x.

Definition A function $F(x)$ is *absolutely continuous* if it can be written

$$F(x) = \int_{-\infty}^{x} f(x)dx.$$

Absolute continuity is stronger than continuity but weaker than differentiability. An example of an absolutely continuous function is one that is:

- continuous everywhere
- differentiable everywhere, except possibly for a countable number of points.

Definition The *probability density function* or pdf, $f_X(x)$, of a continuous random variable X is the function that satisfies

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt \quad \text{for all } x.$$

Notation: We write $X \sim F_X(x)$ for the expression "X has a distribution given by $F_X(x)$" where we read the symbol "$\sim$" as "is distributed as". Similarly, we can write $X \sim f_X(x)$ or$X \sim F_X(x)$, if X and Y have the same distribution, $X \sim Y$.

Theorem A function $f_X(x)$ is a pdf (or pmf) of a random variable X if and only if

1. $f_X(x) \geqslant 0$ for all x.
2. $\int_{-\infty}^{\infty} f_X(x)dx = 1$ (pdf) or $\sum_x f_X(x) = 1$ (pmf).

Example Suppose $\lambda > 0, F(x) = 1 - e^{-\lambda x}$ for $x > 0$ and $F(x) = 0$ otherwise. Is $F(x)$ a cdf? What is the associated pdf?
solution:
$F(x)$ satisfies the three properties of cdf

1. $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$
2. F is nondecreasing: if $a < b$, then $F(a) \leqslant F(b)$
3. F is right-continuous: $\lim_{x\downarrow b} F(x) = F(b)$, or $\lim_{x\to b^+} F(x) = F(b)$.

$F(x)$ is a cdf. Actually, $F(x)$ is the cdf of exponential distribution.

To get the pdf, we only need to differentiate the cdf.

$$f(x) = \frac{dF(x)}{dx} = \begin{cases} \lambda e^{-\lambda x} & \text{for } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Note

- If X is a continuous random variable, then $f(x)$ is not the probability that $X = x$. In fact, if X is an absolutely continuous random variable with density function $f(x)$, then $\Pr(X = x) = 0$. (Why?)
  proof

$$\begin{aligned}\Pr(X = x) &= \lim_{h \to 0} \int_{x-h}^{x+h} f(u)du \\ &= \lim_{h \to 0} F(x+h) - F(x-h) \\ &= F(x+) - F(x-) \\ &= 0\end{aligned}$$

- Because $\Pr(X = a) = 0$, all the following are equivalent:

$$\Pr(a \leqslant X \leqslant b), \quad \Pr(a \leqslant X < b) \quad , \quad \Pr(a < X \leqslant b) \quad \text{and} \quad \Pr(a < X < b)$$

- $f(x)$ can exceed one!

Lecture 13: Sept 18

Last time

- Continuous Random Variables

Today

- Transformations of Random Variables

Transformations of Random Variables

Theorem If X is a r.v. with sample space $\mathcal{X} \subset \mathbb{R}$ and cdf $F_X(x)$, then any function of X, say $Y = g(X)$ is also a random variable. The new random variable Y has a new sample space $\mathcal{Y} = g(X) \subset \mathbb{R}$. The objective is to find the cdf $F_Y(y)$ of Y.

Probability mapping: For any set $A \subset \mathcal{Y}$:

$$\begin{aligned}\Pr(Y \in A) &= \Pr(g(X) \in A) \\ &= \Pr(\{x \in \mathcal{X} : g(x) \in A\}) \\ &= \Pr(X \in g^{-1}(A)),\end{aligned}$$

where we have defined

$$g^{-1}(A) = \{x \in \mathcal{X} : g(x) \in A\}.$$

Notice that $g^{-1}(A)$ is well defined even if $g(\cdot)$ is not necessarily bijective.

Example (Binomial transformation) A discrete random variable X has a *binomial distribution* if its pmf is of the form

$$f_X(x) = \Pr(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \ldots, n,$$

where n is a positive integer and $0 \leqslant p \leqslant 1$. Values such as n and p that can be set to different values, producing different probability distributions, are called *parameters*. Consider a random variable $Y = g(X)$, where $g(x) = n - x$; that is, $Y = n - X$. Here $\mathcal{X} = \{0, 1, \ldots, n\}$ and $\mathcal{Y} = \{y : y = g(x), x \in \mathcal{X}\} = \{0, 1, \ldots, n\}$. For any $y \in \mathcal{Y}$, $n - x = g(x) = y$ if and only if $x = n - y$. Therefore, $g^{-1}(y) = n - y$ and

$$\begin{aligned}f_Y(y) &= \sum_{x \in g^{-1}(y)} f_X(x) \\ &= f_X(n-y) \\ &= \binom{n}{n-y} p^{n-y} (1-p)^{n-(n-y)} \\ &= \binom{n}{y} (1-p)^y p^{n-y}.\end{aligned}$$

Therefore, Y also has a binomial distribution, but with parameters n and $1 - p$.

Example (exercise 2.3) Suppose X has the geometric pmf $f_X(x) = \frac{1}{3}(\frac{2}{3})^x, x = 0, 1, 2, \ldots$. Determine the probability distribution of $Y = X/(X+1)$. Note that here both X and Y are discrete random variables. To specify the probability distribution of Y, specify its pmf. *Solution:*

$$\Pr(Y = y) = \Pr(\frac{X}{X+1} = y) = \Pr(X = \frac{y}{1-y}) = \frac{1}{3}(\frac{2}{3})^{y/(1-y)}, y = 0, \frac{1}{2}, \frac{2}{3}, \ldots, \frac{x}{x+1}, \ldots$$

Lecture 14: Sept 20

Last time

- Transformations of Random Variables

Today

- Transformations of Random Variables

Transformations of Random Variables

Theorem Suppose a continuous random variable X has cdf $F_X(x)$, let $Y = g(X)$, and let $\mathcal{X}$ and $\mathcal{Y}$ be defined as

$$\mathcal{X} = \{x : f(x) > 0\} \quad \text{and} \quad \mathcal{Y} = \{y : y = g(x) \text{ for some } x \in \mathcal{X}\}.$$

Then,

1. If g is an increasing function on $\mathcal{X}$, $F_Y(y) = F_X(g^{-1}(y))$ for $y \in \mathcal{Y}$.
2. If g is a decreasing function on $\mathcal{X}$, $F_Y(y) = 1 - F_X(g^{-1}(y))$ for $y \in \mathcal{Y}$.

Proof: We start with

$$\begin{aligned} F_Y(y) &= \Pr(Y \leqslant y) \\ &= \Pr(g(X) \leqslant y) \end{aligned}$$

1. If g is an increasing function, then $g(X) \leqslant y$ if and only if $X \leqslant g^{-1}(y)$. Therefore, $F_Y(y) = \Pr(g(X) \leqslant y) = \Pr(X \leqslant g^{-1}(y)) = F_X(g^{-1}(y))$.
2. Similarly, if g is a decreasing function, then $g(X) \leqslant y$ if and only if $X \geqslant g^{-1}(y)$. And $F_Y(y) = \Pr(g(X) \leqslant y) = \Pr(X \geqslant g^{-1}(y)) = 1 - F_X(g^{-1}(y))$.

Theorem Let X have pdf $f_X(x)$ and let $Y = g(X)$, where g is a monotone function. Let $\mathcal{X}$ and $\mathcal{Y}$ be defined as

$$\mathcal{X} = \{x : f(x) > 0\} \quad \text{and} \quad \mathcal{Y} = \{y : y = g(x) \text{ for some } x \in \mathcal{X}\}.$$

Suppose that $f_X(x)$ is continuous on $\mathcal{X}$ and that $g^{-1}(y)$ has a continuous derivative on $\mathcal{Y}$. Then the pdf of Y is given by

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y))|\frac{d}{dy}g^{-1}(y)| & y \in \mathcal{Y} \\ 0 & otherwise. \end{cases}$$

Proof:
From last theorem, we have the cdf forms $F_Y(y)$. Then $f_Y(y) = \frac{d}{dy}F_Y(y)$. (finish the proof)
From last theorem, we have

$$F_Y(y) = \begin{cases} F_X(g^{-1}(y)) & \text{if } g \text{ is increasing} \\ 1 - F_X(g^{-1}(y)) & \text{if } g \text{ is decreasing.} \end{cases}$$

We have, by the chain rule,

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \begin{cases} f_X(g^{-1}(y))\frac{d}{dy}g^{-1}(y) & \text{if } g \text{ is increasing} \\ -f_X(g^{-1}(y))\frac{d}{dy}g^{-1}(y) & \text{if } g \text{ is decreasing,} \end{cases}$$

where $\frac{d}{dy}g^{-1}(y) < 0$ when g is decreasing such that $-\frac{d}{dy}g^{-1}(y) = |\frac{d}{dy}g^{-1}(y)|$.

Example (Square transformation) Suppose X is a continuous random variable. For $y > 0$, the cdf of $Y = X^2$ is

$$F_Y(y) = \Pr(Y \leqslant y) = \Pr(X^2 \leqslant y) = \Pr(-\sqrt{y} \leqslant X \leqslant \sqrt{y}).$$

Because x is continuous, we can drop the equality from the left endpoint and obtain

$$\begin{aligned} F_Y(y) &= \Pr(-\sqrt{y} < X \leqslant \sqrt{y}) \\ &= \Pr(X \leqslant \sqrt{y}) - \Pr(X \leqslant -\sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y}). \end{aligned}$$

The pdf of Y can now be obtained from the cdf by differentiation:

$$\begin{aligned} f_Y(y) &= \frac{d}{dy}F_Y(y) \\ &= \frac{d}{dy}\left[F_X(\sqrt{y}) - F_X(-\sqrt{y})\right] \\ &= \frac{1}{2\sqrt{y}}f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}}f_X(-\sqrt{y}), \end{aligned}$$

where we use the chain rule to differentiate $F_X(\sqrt{y})$ and $F_X(-\sqrt{y})$.

Lecture 15: Sept 23

Last time

- Transformations of Random Variables

Today

- One-page one-sided letter-size cheat sheet for midterm 1
- Transformations of Random Variables
- Expected Values

Transformations of Random Variables

Example (Linear transformation) Suppose X is a continuous random variable with pdf $f_X(x)$. Let

$$Y = a + bX, \quad \frac{dy}{dx} = b.$$

Then

$$f_Y(y) = f_X\left[g^{-1}(y)\right] |\frac{dx}{dy}| = f_X(\frac{y-a}{b})\frac{1}{|b|}.$$

This transformation is often used when X has mean 0 and standard deviation 1. The linear transformation above creates a random variable Y with a distribution that has the same shape as that of X but has mean a and variance b^2.

Conversely, if Y has mean a and standard deviation b, then $X = (Y-a)/b$ has mean 0 and standard deviation 1. This is called sometimes the "Studentized" transformation.

Example (Normal distribution) Let $X \sim N(0,1)$:

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad -\infty < x < \infty.$$

The transformation

$$Y = \mu + \sigma X, \quad X = \frac{Y-\mu}{\sigma}$$

yields

$$f_Y(y) = f_X(\frac{y-\mu}{\sigma})\frac{1}{\sigma} = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-\mu)^2}{2\sigma^2}}.$$

More generally, a distribution is a member of the class of *location-scale* distributions if the distribution of a linear transformation of a random variable with that distribution has the same distribution, but with different parameters.

Example (Square root of an exponential RV) Suppose $X \sim exp(\lambda)$, so that

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geqslant 0 \\ 0 & otherwise \end{cases}$$

and consider the distribution of $Y = \sqrt{X}$. The transformation

$$y = g(x) = \sqrt{x}, \quad x \geqslant 0$$

is one-to-one and has an inverse $x = y^2$ with $dx/dy = 2y$. Thus

$$f_Y(y) = f_X(y^2)2y = 2\lambda y e^{-\lambda y^2}, \quad y \geqslant 0.$$

This distribution is a particular form of the Rayleigh distribution and is a special case of the Weibull distribution.

Theorem (Probability integral transformation) Let X have continuous cdf $F_X(x)$ and define the random variable Y as $Y = F_X(X)$. Then Y is uniformly distributed on $(0, 1)$, that is, $\Pr(Y \leqslant y) = y, 0 < y < 1$.

Before we prove this theorem, we will digress for a moment and look at F_X^{-1}, the inverse of the cdf F_X, in some detail. If F_X is strictly increasing, then F_X^{-1} is well defined by

$$F_X^{-1}(y) = x \iff F_X(x) = y.$$

However, if F_X is constant on some interval, then F_X^{-1} is not well defined as Figure 14.1 illustrates. Any $x_1 \leqslant x \leqslant x_2$ satisfies $F_X(x) = y$

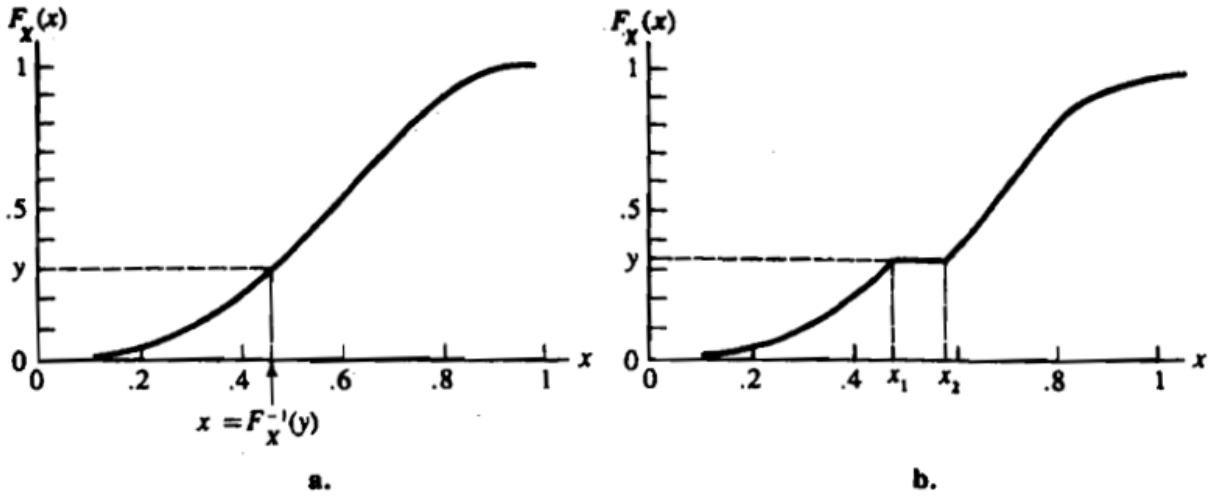

Figure 14.1: Figure 2.1.2. (a) $F_X(x)$ strictly increasing; (b) $F_X(x)$ nondecreasing

This problem is avoided by defining F_X^{-1} for $0 < y < 1$ by

$$F_X^{-1}(y) = \inf\{x : F_X(x) \geqslant y\}.$$

With this definition, for Figure 14.1(b), we have $F_X^{-1}(y) = x_1$.
Proof:
For $Y = F_X(X)$, we have, for $0 < y < 1$,

$$\begin{aligned}
\Pr(Y \leqslant y) &= \Pr(F_X(X) \leqslant y) \\
&= \Pr(F_X^{-1}[F_X(X)] \leqslant F_X^{-1}(y)) \quad (F_X^{-1} \text{ is increasing}) \\
&= \Pr(X \leqslant F_X^{-1}(y)) \\
&= F_X(F_X^{-1}(y)) \qquad (\text{definition of } F_X) \\
&= y.
\end{aligned}$$

One application of the probability integral transformation is in the generation of random samples from a particular distribution. If it is required to generate an observation X from a population with cdf F_X, we need only generate a uniform random number U, between 0 and 1, and solve for x in the equation $F_X(x) = u$.

Lecture 16: Sept 25

Last time

- Transformations of Random Variables

Today

- One-page one-sided letter-size cheat sheet for midterm 1
- Expected Values
- Moments

Expected Values

Definition The *expected value* or *mean* of a random variable $g(X)$, denoted by $Eg(X)$, is

$$Eg(X) = \begin{cases} \int_{-\infty}^{\infty} g(x)f(x)dx & \text{if } X \text{ is continuous} \\ \sum_{x\in\mathcal{X}} g(x)\Pr(X=x) & \text{if } X \text{ is discrete} \end{cases}$$

Provided the integral or summation exists.

If we let $g(X) = X$, then we get

$$EX = \begin{cases} \int_{-\infty}^{\infty} xf(x)dx & \text{if } X \text{ is continuous} \\ \sum_{x\in\mathcal{X}} x\Pr(X=x) & \text{if } X \text{ is discrete} \end{cases}$$

Example (Exponential mean) Suppose X has an *exponential distribution* with parameter λ, $X \sim Exp(\lambda)$, that is, it has pdf given by

$$f_X(x) = \frac{1}{\lambda}e^{-x/\lambda}, \quad 0 \leqslant x < \infty, \lambda > 0.$$

Find out EX.
Solution:

$$\begin{aligned} EX &= \int_0^{\infty} \frac{1}{\lambda} x e^{-x/\lambda} dx \\ &= -xe^{-x/\lambda}\Big|_0^{\infty} + \int_0^{\infty} e^{-x/\lambda} dx \\ &= \int_0^{\infty} e^{-x/\lambda} dx \\ &= \lambda \end{aligned}$$

Example (Binomial mean) if X has a *binomial distribution*, $X \sim Binomial(n, p)$, its pmf is given by

$$\Pr(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \ldots, n,$$

where n is a positive integer, $0 \leqslant p \leqslant 1$, and for every fixed pair n and p the pmf sums to 1. Find out EX.
Solution:

$$EX = \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x}, = \sum_{x=1}^{n} x \binom{n}{x} p^x (1-p)^{n-x}.$$

Using the identity $x\binom{n}{x} = n\binom{n-1}{x-1}$, we have

$$\begin{aligned} EX &= \sum_{x=1}^{n} n \binom{n-1}{x-1} p^x (1-p)^{n-x} \\ &= \sum_{y=0}^{n-1} n \binom{n-1}{y} p^{y+1} (1-p)^{n-(y+1)} \\ &= np \sum_{y=0}^{n-1} \binom{n-1}{y} p^y (1-p)^{n-1-y} \\ &= np, \end{aligned}$$

since the last summation must be 1, being the sum over all possible values of a binomial$(n-1, p)$ pmf.

Lecture 17: Sept 27

Last time

- Exam 1 covers up to today's lecture
- Expectations (2.2)

Today

- Expectations (2.2)
- Moments and moment generation function

The process of taking expectations is a linear operation, which means that the expectation of a linear function of X can be easily evaluated by noting that for any constants a and b, such that

$$E(aX + b) = aEX + b$$

Theorem Let X be a random variable and let a, b, and c be constants. Then for any functions $g_1(x)$ and $g_2(x)$ whose expectations exist,

1. $E(ag_1(X) + bg_2(X) + c) = aEg_1(X) + bEg_2(X) + c$.
2. If $g_1(x) \geqslant 0$ for all x, then $Eg_1(X) \geqslant 0$.
3. If $g_1(x) \geqslant g_2(x)$ for all x, then $Eg_1(X) \geqslant Eg_2(X)$.
4. If $a \leqslant g_1(x) \leqslant b$ for all x, then $a \leqslant Eg_1(X) \leqslant b$.

Proof:
We will give details for only the continuous case, the discrete case being similar. By definition

$$\begin{aligned} E(ag_1(X) + bg_2(X) + c) &= \int_{-\infty}^{\infty} [ag_1(x) + bg_2(x) + c] f_X(x)dx \\ &= \int_{-\infty}^{\infty} ag_1(x)f_X(x)dx + \int_{-\infty}^{\infty} bg_2(x)f_X(x)dx + \int_{-\infty}^{\infty} cf_X(x)dx \\ &= aEg_1(X) + bEg_2(X) + c \end{aligned}$$

The other three properties are proved in a similar manner (shown in class).

Example (Method of indicators) An example of how the above properties are useful. Let $X \sim Binomial(n, p)$ for n positive integer and $0 \leqslant p \leqslant 1$ (n is the number of independent identical binary trials and p is the probability of success). We can write

$$X = \sum_{i=1}^{n} I_i$$

where I_i is the indicator that i^{th} trial is a success (i.e. $I_i \overset{\text{i.i.d.}}{\sim} Bernoulli(p)$). We have

$$EI_i = 1 \cdot p + 0 \cdot (1-p) = p.$$

Therefore,

$$EX = \sum_{i=1}^{n} EI_i = \sum_{i=1}^{n} p = np.$$

Theorem For a non-negative random variable X (i.e., $f(x) = 0$ for $x < 0$).

$$EX = \begin{cases} \int_0^\infty (1 - F(x))dx, & X \text{ continuous} \\ \sum_{x=0}^\infty (1 - F(x)), & X \text{ discrete, and } S_X = \{0, 1, 2, \dots\} \end{cases}$$

Proof:
We prove the continuous case first,

$$\begin{aligned} \int_0^\infty [1 - F(x)]\, dx &= \int_0^\infty [1 - \Pr(X \leqslant x)]\, dx \\ &= \int_0^\infty \Pr(X > x) dx \\ &= \int_0^\infty \int_{y=x}^\infty f_X(y) dy dx \\ &= \int_0^\infty \int_{x=0}^y f_X(y) dx dy \\ &= \int_0^\infty y f_X(y) dy \\ &= EX. \end{aligned}$$

Then, for discrete case, we have

$$\begin{aligned} \sum_{x=0}^\infty (1 - F(x)) &= \sum_{x=0}^\infty \Pr(X > x) \\ &= \sum_{x=0}^\infty \sum_{y=x+1}^\infty \Pr(X = y) \\ &= \sum_{y=1}^\infty \sum_{x=0}^{y-1} \Pr(X = y) \\ &= \sum_{y=1}^\infty y \Pr(X = y) \\ &= EX \end{aligned}$$

Lecture 18: Oct 7

Last time

- Midterm Exam 1

Today

- Midterm 1 review
- Moments and moment generating function

Moments

Example (Minimizing distance) The expected value of a random variable has another property, one that we can think of as relating to the interpretation of EX as a good guess at a value of X.

Suppose we measure the distance between a random variable X and a constant b by $(X-b)^2$. The closer b is to X, the smaller this quantity is. We can now determine the value of b that minimizes $E\left[(X-b)^2\right]$ and, hence, will provide us with a good predictor of X. (Note that it does no good to look for a value of b that minimizes $(X-b)^2$, since the answer would depend on X, making it a useless predictor of X.)

We could proceed with the minimization of $E(X-b)^2$ by using calculus, but there is a simpler method:

$$\begin{aligned} E(X-b)^2 &= E(X-EX+EX-b)^2 \\ &= E\left[(X-EX)+(EX-b)\right]^2 \\ &= E(X-EX)^2+(EX-b)^2+2E\left[(X-EX)(EX-b)\right], \end{aligned}$$

where we have expanded the square. Note that $E\left[(X-EX)(EX-b)\right] = (EX-b)E(X-EX) = 0$, since $EX-b$ is constant and comes out of the expectation, $E(X-EX) = EX-EX = 0$. This means

$$E(X-b)^2 = E(X-EX)^2+(EX-b)^2.$$

Such that $E(X-b)^2$ is minimized at $b = EX$. And $E(X-EX)^2$ is actually the variance of X ($VarX = E(X-EX)^2$).

The various moments of a distribution are an important class of expectations.

Definition For each integer n, the nth *moment* of X (or $F_X(x)$), μ_n', is

$$\mu_n' = EX^n.$$

The nth *central moment* of X, μ_n, is

$$\mu_n = E(X-\mu)^n,$$

where $\mu = \mu_1' = EX$.

Notes:

- $\mu_0' = EX^0 = 1$
- μ_1' is the *mean*, usually denoted by μ.
- $\mu_0 = E(X - \mu)^0 = 1$
- $\mu_1 = 0$
- $\mu_2 = E(X - EX)^2$ is the *variance*
- $\mu_3 = E(X - EX)^3$ is related to the *skewness*.
- $\mu_4 = E(X - EX)^4$ is related to the *kurtosis*.

Definition The *variance* of a random variable X is its second central moment, $\text{Var}(X) = E[(X - EX)^2]$. The positive square root of $\text{Var}(X)$ is the *standard deviation* of X.

The variance gives a measure of the degree of spread of a distribution around its mean. Figure 29.4 shows a plot of two samples, one sample draws 100 numbers from a normal distribution with mean 0 and variance 1, $N(0, 1)$. The other sample draws 100 numbers from a normal distribution with mean 0 and variance 100, $N(0, 100)$.

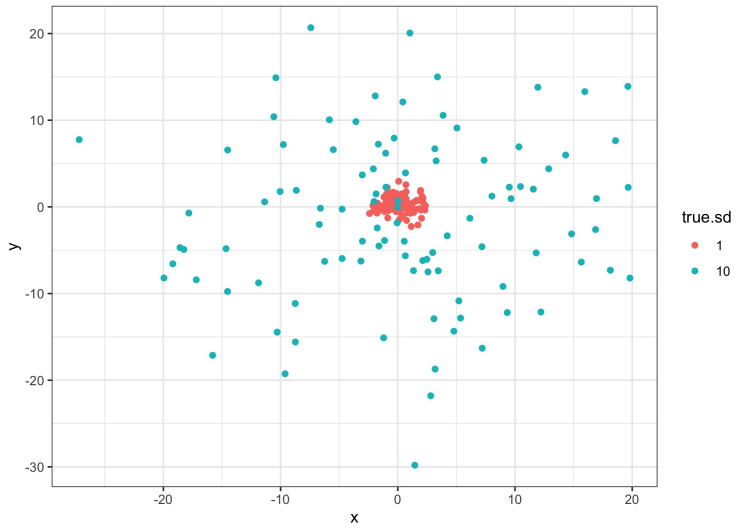

Figure 18.2: Figure 2.1.2. Two samples of 100 numbers drawn from $N(0, 1)$ and $N(0, 100)$.

Example (Exponential variance) Let X have the exponential(λ) distribution. We can calculate the variance of X now.
Solution:

$$\begin{aligned}\mathrm{Var}\,(X) &= E(X-\lambda)^2 \\ &= \int_0^\infty (x-\lambda)^2 \frac{1}{\lambda} e^{-x/\lambda} dx\end{aligned}$$

$$\begin{aligned}\mathrm{Var}\,(X) &= \int_0^\infty (x^2 - 2x\lambda + \lambda^2)\frac{1}{\lambda} e^{-x/\lambda} dx \\ &= \int_0^\infty x^2 \frac{1}{\lambda} e^{-x/\lambda} dx - 2\int_0^\infty x\lambda \frac{1}{\lambda} e^{-x/\lambda} dx + \lambda^2 \\ &= EX^2 - \lambda^2 \\ &= \lambda^2\end{aligned}$$

Theorem If X is a random variable with finite variance, then for any constants a and b,

$$\mathrm{Var}\,(aX+b) = a^2 \mathrm{Var}\,(X)\,.$$

Proof:
From the definition, we have

$$\begin{aligned}\mathrm{Var}\,(aX+b) &= E\left[(aX+b) - E(aX+b)\right]^2 \\ &= E(aX - aEX)^2 \\ &= a^2 E(X-EX)^2 \\ &= a^2 \mathrm{Var}\,(X)\,.\end{aligned}$$

It is sometimes to use an alternative formula for the variance, given by

$$\mathrm{Var}\,(X) = E(X^2) - (EX)^2,$$

which is easily established by

$$\begin{aligned}\mathrm{Var}\,(X) &= E(X-EX)^2 = E\left[X^2 - 2XEX + (EX)^2\right] \\ &= EX^2 - 2(EX)^2 + (EX)^2 \\ &= EX^2 - (EX)^2.\end{aligned}$$

Lecture 19: Oct 9

Last time

- Moments

Today

- Midterm exam 1 review
- Internal midterm evaluation open
- Vote for alternative weighting scheme
- Moment generating function

Example (Binomial variance) Let $X \sim Binomial(n, p)$, that is ,

$$\Pr(X = x) = \binom{n}{x} p^x (1-p)^{n-x}.$$

What is the variance of X?
Solutions:
Method #1:
We want to find EX^2 first. We use the

$$EX^2 = \sum_{x=0}^{n} x^2 \binom{n}{x} p^x (1-p)^{n-x}.$$

we use the same property $x^2\binom{n}{x} = xn\binom{n-1}{x-1}$. We then have

$$\begin{aligned}
EX^2 &= n \sum_{x=1}^{n} x \binom{n-1}{x-1} p^x (1-p)^{n-x} \\
&= n \sum_{y=0}^{n-1} (y+1) \binom{n-1}{y} p^{y+1} (1-p)^{n-1-y} \\
&= np \sum_{y=0}^{n-1} y \binom{n-1}{y} p^y (1-p)^{n-1-y} + np \sum_{y=0}^{n-1} \binom{n-1}{y} p^y (1-p)^{n-1-y} \\
&= np \cdot (n-1)p + np \\
&= n(n-1)p^2 + np.
\end{aligned}$$

And now

$$\begin{aligned}
\mathrm{Var}\,(X) &= EX^2 - (EX)^2 \\
&= n(n-1)p^2 + np - (np)^2 \\
&= np - np^2 \\
&= np(1-p).
\end{aligned}$$

Method #2:
Recall that we could write $X = \sum_{i=1}^{n} I_i$, where $I_i \overset{\text{i.i.d.}}{\sim} Bernoulli(p)$. Then

$$\begin{aligned}
\text{Var}\,(X) &= \text{Var}\left(\sum_{i=1}^{n} I_i\right) \\
&= \sum_{i=0}^{n} \text{Var}\,(I_i) \qquad (I_i\text{'s are independent}) \\
&= n\text{Var}\,(I_i) \qquad (I_i\text{'s are identically distributed}) \\
&= n\left[E(I_i^2) - (EI_i)^2\right] \\
&= n\left[p - p^2\right] \\
&= np(1-p).
\end{aligned}$$

Definition Let X be a random variable with cdf F_X. The *moment generating function (mgf)* of X (or F_X), denoted by $M_X(t)$, is

$$M_X(t) = Ee^{tX},$$

provided that the expectation exists for t in some neighborhood of 0. That is, there is an $h > 0$ such that, for all t in $-h < t < h$, Ee^{tX} exists. If the expectation does not exist in a neighborhood of 0, we say that the moment generating function does not exist.

More explicitly, we can write the mgf of X as

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x)dx, \quad \text{if } X \text{ is continuous,}$$

or

$$M_X(t) = \sum_{x} e^{tx}\,\text{Pr}(X = x), \quad \text{if } X \text{ is discrete.}$$

It is easy to see how the mgf generates moments as in the following theorem.

Lecture 20: Oct 11

Last time

- Presentations
- Moment generating function

Today

- Internal midterm evaluation open
- Moment generating function
- Common Discrete Distributions (Chapter 3)

Definition Let X be a random variable with cdf F_X. The *moment generating function (mgf)* of X (or F_X), denoted by $M_X(t)$, is

$$M_X(t) = Ee^{tX},$$

provided that the expectation exists for t in some neighborhood of 0. That is, there is an $h > 0$ such that, for all t in $-h < t < h$, Ee^{tX} exists. If the expectation does not exist in a neighborhood of 0, we say that the moment generating function does not exist.

More explicitly, we can write the mgf of X as

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x)dx, \quad \text{if } X \text{ is continuous,}$$

or

$$M_X(t) = \sum_x e^{tx} \Pr(X = x), \quad \text{if } X \text{ is discrete.}$$

It is easy to see how the mgf generates moments as in the following theorem.

Theorem If X has mgf $M_X(t)$, then

$$EX^n = M_X^{(n)}(0),$$

where we define

$$M_X^{(n)}(0) = \frac{d^n}{dt^n} M_X(t)\bigg|_{t=0}.$$

That is, the n^{th} moment is equal to the n^{th} derivative of $M_X(t)$ evaluated at $t = 0$.

Proof:

$$\begin{aligned}\frac{d}{dt}M_X(t) &= \frac{d}{dt}\int_{-\infty}^{\infty} e^{tx} f_X(x)dx \\ &= \int_{-\infty}^{\infty} \left(\frac{d}{dt}e^{tx}\right) f_X(x)dx \\ &= \int_{-\infty}^{\infty} \left(xe^{tx}\right) f_X(x)dx \\ &= E(Xe^{tX}).\end{aligned}$$

Therefore,

$$\frac{d}{dt}M_X(t)\bigg|_{t=0} = E(Xe^{tX})\bigg|_{t=0} = EX.$$

Proceeding in an analogous manner, we can establish that

$$\frac{d^n}{dt^n}M_X(t)\bigg|_{t=0} = E(X^n e^{tX})\bigg|_{t=0} = EX^n.$$

Example (Binomial mgf) Let $X \sim Binomial(n, p)$, then its mgf is

$$\begin{aligned}M_X(t) &= \sum_{x=0}^{n} e^{tx}\binom{n}{x}p^x(1-p)^{n-x} \\ &= \sum_{x=0}^{n}\binom{n}{x}(pe^t)^x(1-p)^{n-x} \\ &= [pe^t + (1-p)]^n.\end{aligned}$$

Lecture 21: Oct 14

Last time

- Internal midterm evaluation open
- Moment generating function

Today

- Moment generating function
- Common Discrete Distributions (Chapter 3)

Theorem Let $F_X(x)$ and $F_Y(y)$ be two cdfs all of whose moments exist.

1. If X and Y have **bounded support**, then $F_X(u) = F_Y(u)$ for all u if and only if $EX^r = EY^r$ for all integers $r = 0, 1, 2, \ldots$.
2. If the moment generating functions exist and $M_X(t) = M_Y(t)$ for all t in some neighborhood of 0, then $F_X(u) = F_Y(u)$ for all u.

Theorem (Convergence of mgfs) Suppose $\{X_i, i = 1, 2, \ldots\}$ is a sequence of random variables, each with mgf $M_{X_i}(t)$. Furthermore, suppose that

$$\lim_{i\to\infty} M_{X_i}(t) = M_X(t), \qquad \text{for all } t \text{ in a neighborhood of } 0,$$

and $M_X(t)$ is an mgf. Then there is a unique cdf F_X whose moments are determined by $M_X(t)$ and, for all x where $F_X(x)$ is continuous, we have

$$\lim_{i\to\infty} F_{X_i}(x) = F_X(x).$$

That is, *convergence*, for $|t| < h$, of mgfs to an mgf implies *convergence* of cdfs.

Poisson approximation One approximation that is usually taught in elementary statistics courses is that binomial probabilities can be approximated by Poisson probabilities. It is taught that the Poisson approximation is valid "when n is large and np is small", and rules of thumb are sometimes given.

The $Poisson(\lambda)$ pmf is given by

$$\Pr(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, 2, \ldots,$$

where λ is a positive constant. The approximation states that if $X \sim Binomial(n, p)$ and $Y \sim Poisson(\lambda)$, with $\lambda = np$, then

$$\Pr(X = x) \approx \Pr(Y = x)$$

for large n and small np. We now show that the mgf converge, lending credence to this approximation. Recall that

$$M_X(t) = [pe^t + (1-p)]^n.$$

For the $Poisson(\lambda)$ distribution, we can calculate (HW4, exercise 2.33)

$$M_Y(t) = e^{\lambda(e^t-1)},$$

and if we define $p = \lambda/n$, then $M_X(t) = [1 + (e^t - 1)\lambda/n]^n$ such that $M_X(t) \to M_Y(t)$ as $n \to \infty$.

Theorem For any constant a and b, the mgf of the random variable $aX + b$ is given by

$$M_{aX+b} = e^{bt} M_X(at).$$

Proof:
By definition,

$$\begin{aligned} M_{aX+b} &= E\left(e^{(aX+b)t}\right) \\ &= E\left(e^{(aX)t} e^{bt}\right) \\ &= e^{bt} E\left(e^{(aX)t}\right) \\ &= e^{bt} M_X(at). \end{aligned}$$

Common Discrete Distribution

Why parametric models?

- *Parametric models* or *distribution families* have a specific form but can change according to a fixed number of parameters.
- The objective is to model a population. Parametric models are often appropriate in common situations with similar mechanisms.
- Parametric models have many known and useful properties and are easy to work with. When fitting a population, only a few parameters need to be estimated: *parametric inference*.
- Sometimes one does not want to make parametric assumptions and would rather work with non-parametric models. But non-parametric models can be infinite dimensional.
- In this course, we emphasize parametric models.

Discrete uniform X has the discrete unifrom$(1, N)$ distribution if X is equally likely to be one of $\{1, 2, \ldots, N\}$.

- Sample space: $\{1, 2, \ldots, N\}$
- pmf:

$$f_X(x) = \frac{1}{N}, \quad x = 1, 2, \ldots, N$$

- cdf:

$$F_X(x) = \Pr(X \leqslant x) = \begin{cases} 0 & x < 0 \\ \lfloor x \rfloor / N & 0 \leqslant x < N \\ 1 & N \leqslant x \end{cases}$$

- moments:

$$EX = \frac{N+1}{2}$$

Lecture 22: Oct 16

Last time

- Common Discrete Distributions (Chapter 3)

Today

- Common Discrete Distributions (Chapter 3)

Bernoulli Distribution Consider an experiment where outcomes are binary (say, Success or Failure) and the probability of success is p. Define the following random variable

$$Y = \begin{cases} 1 & \text{outcome is success} \\ 0 & \text{outcome is failure} \end{cases}$$

Then, Y has a Bernoulli Distribution.

- Sample space: $\{0, 1\}$.
- pmf: $\Pr(Y = 1) = p$ and $\Pr(Y = 0) = 1 - p$. We can write this as:

$$f(y) = \Pr(Y = y) = \begin{cases} p^y(1-p)^{1-y} & y = 0, 1 \\ 0 & othersie \end{cases}$$

- what are the cdf, mean and variance?

Binomial Distribution A $Binomial(n, p)$ random variable X is defined as the number of successes in n i.i.d. (independent, identically distributed) Bernoulli trials, each with probability p of success:

$$X = \sum_{i=1}^{n} Y_i, \quad Y_1, \ldots, Y_n \overset{\text{i.i.d.}}{\sim} Bernoulli(p)$$

- Sample space: $\{0, 1, \ldots, n\}$
- pmf:

$$f_X(s) = \begin{cases} \binom{n}{s} p^x (1-p)^{n-s} & s = 0, 1, \ldots, n \\ 0 & otherwise \end{cases}$$

- cdf:

$$F_X(x) = \sum_{s=0}^{x} \binom{n}{s} p^s (1-p)^{n-s} \quad \text{(no closed form)}$$

- mean:

$$EX = np$$

- variance:

$$Var(X) = np(1-p)$$

Poisson Distribution The Poisson distribution was derived by the French mathematician Poisson in 1837 as a limiting version of the binomial distribution. The Poisson distribution is often used to model the number of occurrences in a given time interval. One of the basic assumptions on which the Poisson distribution is built is that, for small time intervals, the probability of an arrival is proportional to the length of waiting time. This makes it a reasonable model for situations such as waiting for a bus, waiting for customers to arrive in a bank.

The Poisson distribution has a single parameter λ, sometimes called the intensity parameter. A Poisson random variable X, takes values in the nonnegative integers with pmf

$$\Pr(X = x|\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, \ldots$$

To see that $\sum_{x=0}^{\infty} P(X = x|\lambda) = 1$, recall the Taylor series expansion of $e^{\lambda} = \sum_{i=0}^{\infty} \frac{\lambda^i}{i!}$. Thus

$$\sum_{x=0}^{\infty} \Pr(X = x|\lambda) = e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{-\lambda}e^{\lambda} = 1$$

What is the mean and variance of X?

$$\begin{aligned} EX &= \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!} \\ &= \sum_{x=1}^{\infty} \frac{e^{-\lambda}\lambda^x}{(x-1)!} \\ &= \lambda \sum_{x=1}^{\infty} \frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!} \\ &= \lambda \sum_{y=0}^{\infty} \frac{e^{-\lambda}\lambda^y}{y!} \\ &= \lambda \end{aligned}$$

Similarly

$$\begin{aligned} EX^2 &= \sum_{x=0}^{\infty} x^2 \frac{e^{-\lambda}\lambda^x}{x!} \\ &= \sum_{x=1}^{\infty} x \frac{e^{-\lambda}\lambda^x}{(x-1)!} \\ &= \sum_{x=1}^{\infty} \frac{e^{-\lambda}\lambda^x}{(x-1)!} + \sum_{x=2}^{\infty} \frac{e^{-\lambda}\lambda^x}{(x-2)!} \\ &= \lambda + \lambda^2 \end{aligned}$$

So that

$$Var(X) = EX^2 - (EX)^2 = \lambda$$

- Sample space: $\{0, 1, \dots\}$
- pmf: $\Pr(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}$
- cdf: $F_X(x) = \sum_{s=0}^{x} \frac{e^{-\lambda}\lambda^s}{s!}$

Lecture 23: Oct 18

Last time

- Common Discrete Distributions (Chapter 3)

Today

- Common Discrete Distributions (Chapter 3)
- Common Continuous Distributions

Hypergeometric Distribution Suppose a population of N entities is made up of two types: M of the first type and $N-M$ of the second type. Suppose we take a sample of size K. We wish to know X, the number in the sample of the first type. The probability mass function of X is given by:

$$f_X(x) = \frac{\binom{M}{x}\binom{N-M}{k-x}}{\binom{N}{k}}$$

for $x = \max(0, M-N+K), \ldots, \min(M, K)$.

The sample space is defined so that all binomial coefficients are valid. We must have:

$$0 \leqslant x \leqslant K, \quad 0 \leqslant x \leqslant M, \quad 0 \leqslant K-x \leqslant N-M$$

Often $K < M$ and $K < N-M$ so the range becomes $0 \leqslant x \leqslant K$.

Hypergeometric vs Binomial We can show that the limiting form of the hypergeometric pmf is the binomial pmf

$$\begin{aligned}
\Pr(s) &= \frac{\binom{M}{s}\binom{N-M}{n-s}}{\binom{N}{n}} \\
&= \frac{\frac{M!}{s!(M-s)!}\frac{(N-M)!}{(n-s)!(N-M-n+s)!}}{\frac{N!}{n!(N-n)!}} \\
&= \frac{\frac{n!}{s!(n-s)!}\frac{M!}{(M-s)!}\frac{(N-M)!}{(N-M-n+s)!}}{\frac{N!}{(N-n)!}}
\end{aligned}$$

Note

$$\begin{aligned}
\frac{M!}{(M-s)!} &= \frac{M(M-1)(M-2)\dots(M-s)!}{(M-s)!} \\
&= M^s\left[1(1-\frac{1}{M})\dots(1-\frac{s-1}{M})\right] \\
\frac{N!}{(N-n)!} &= N^n\left[1(1-\frac{1}{N})\dots(1-\frac{n-1}{N})\right] \\
\frac{(N-M)!}{[(N-M)-(n-s)]!} &= (N-M)^{n-s}\left[1(1-\frac{1}{N-M})\dots(1-\frac{n-s-1}{N-M})\right]
\end{aligned}$$

Letting $N \to \infty, M \to \infty, \frac{M}{N} \to p$, we have

$$\begin{aligned}
\Pr(s) &= \frac{\binom{M}{s}\binom{N-M}{n-s}}{\binom{N}{s}} \\
&\approx \binom{n}{s}\frac{M^s(N-M)^{n-s}}{N^n} \\
&= \binom{n}{s}\left(\frac{M}{N}\right)^s\left(1-\frac{M}{N}\right)^{n-s} \\
&\to \binom{n}{s}p^s(1-p)^{n-s}
\end{aligned}$$

In summary, we have

Hypergeometric $\to$	Binomial $\to$	Poisson
$N \to \infty$	$n \to \infty$	$\lambda = np$
$M \to \infty$	$p \to 0$	
$\frac{M}{N} \to p$	$np \to \lambda$	

Lecture 24: Oct 21

Last time

- Common Discrete Distributions (Chapter 3)

Today

- Negative binomial distribution
- Common Continuous Distributions

Geometric Distribution Consider a series of iid Bernoulli Trials with p = probability of success in each trial. Define a random variable X representing the number of trials until first success. Note X includes the trial at which the success occurs (one parameterization). Then, X has a geometric distribution.

- Sample space: $\{1, 2, \dots\}$
- pmf:
$$f(x) = \Pr(X = x) = \begin{cases} p(1-p)^{x-1} & x = 1, 2, \dots \\ 0 & otherwise \end{cases}$$
- cdf:
$$F(x) = \Pr(X \leqslant x) = 1 - (1-p)^x$$
- Moments:
$$E(X) = 1/p$$
$$Var(X) = (1-p)/p^2$$

Memoryless property. Suppose $k > i$, then

$$\Pr(X > k | X > i) = \Pr(X > k - i)$$

Proof:

$$\begin{aligned} \Pr(X > k | X > i) &= \frac{\Pr(X > k)}{\Pr(X > i)} = \frac{(1-p)^k}{(1-p)^i} \\ &= (1-p)^{k-i} = \Pr(X > k - i) \end{aligned}$$

Example Suppose X is number of years you live, and X follows a geometric distribution, then

$$\begin{aligned} \Pr(\text{survive two more years}) &= \Pr(X > \text{current age} + 2 | X > \text{current age}) \\ &= \Pr(X > 2) \end{aligned}$$

This model is clearly too simple for human populations (since we do age).

Negative Binomial Distribution Still in the context of iid Bernoulli trials, define a random variable corresponding to the number of trials required to have s successes. We say $X \sim Negbin(s, p)$.

- Sample space: $\{s, (s+1), \dots\}$
- pmf: for $x = s, s+1, s+2, \dots$

$$\begin{aligned} f(x) &= \binom{x-1}{s-1} p^{s-1} q^{x-s} \cdot p \\ &= \binom{x-1}{s-1} p^s q^{x-s} \end{aligned}$$

- cdf: no closed form
- Expectation: $EX = s/p$.
- Variance: $Var(X) = s(1-p)/p^2$

Notes

- Why the name? See Casella & Berger p.95.
- $X \sim Negbin(1, p)$ is the same as $X \sim Geometric(p)$
- $Negbin(n, p)$ is the same as the sum of n $Geometric(p)$ random variables

Other parameterizations The negative binomial distribution is sometimes defined in terms of the random variable Y = number of failures before the rth success. Then

- Sample space: $\{0, 1, 2, \dots\}$
- pmf

$$f(y) = \binom{r+y-1}{y} p^r q^y, \quad y = 0, 1, 2, \dots$$

- cdf: no closed form
- Expectation: $EY = r(1-p)/p$
- Variance: $Var(Y) = r(1-p)/p^2$

Negarive binomial vs. Poisson The negative binomial distribution is often good for modeling count data as an alternative to the Poisson. In the previous parameterization, define

$$\lambda = \frac{r(1-p)}{p} \iff p = \frac{r}{r+\lambda}$$

Then we have

$$\begin{aligned} EX &= \lambda \\ Var(X) &= \frac{\lambda}{p} = \lambda(1 + \frac{\lambda}{r}) = \lambda + \frac{\lambda^2}{r} \end{aligned}$$

For the Poisson we had that the variance equals the mean.
For the negative binomial, the variance is equal to the mean plus a quadratic term. Thus the negative binomial can capture overdispersion in count data.
In the previous parameterization, the pmf becomes

$$\begin{aligned} f(y) &= \binom{r+y-1}{y} p^r q^y = \frac{(r+y-1)!}{y!(r-1)!} \left(\frac{r}{r+\lambda}\right)^r \left(\frac{\lambda}{r+\lambda}\right)^y \\ &= \frac{\lambda^y}{y!} \frac{r(r+1)\dots(r+y-1)}{(r+\lambda)^y} \left(1+\frac{\lambda}{r}\right)^{-r} \end{aligned}$$

Letting $r \to \infty$, we get

$$f(x) \to \frac{\lambda^x}{x!} e^{-\lambda}$$

So for large r, the negative binomial can be approximated by a Poisson with parameter $\lambda = r(1-p)/p$.

Lecture 25: Oct 23

Last time

- Common Continuous Distributions

Today

- Common Continuous Distributions

Common continuous distributions

Uniform Distribution A random variable X having a pdf

$$f(x) = \begin{cases} 1 & \text{for } 0 < x \leqslant 1 \\ 0 & otherwise \end{cases}$$

is said to have a *uniform distribution* over the interval $(0, 1)$.

The cdf is:

$$F(y) = \int_{-\infty}^{y} f(x)dx = \begin{cases} 0 & \text{for } y \leqslant 0 \\ y & \text{for } 0 \leqslant y \leqslant 1 \\ 1 & \text{for } y > 1 \end{cases}$$

- Unifrom; $Y \sim U[a, b]$
- sample space: $[a, b]$
- pdf:

$$f(y) = \begin{cases} \frac{1}{b-a} & \text{for } a < y \leqslant b \\ 0 & otherwise \end{cases}$$

- cdf:

$$F(y) = \int_{-\infty}^{y} f(x)dx = \begin{cases} 0 & \text{for } y \leqslant a \\ \frac{y-a}{b-a} & \text{for } a \leqslant y \leqslant b \\ 1 & \text{for } y > b \end{cases}$$

- moments:

$$E(Y) = (a + b)/2$$
$$Var(Y) = \frac{(b-a)^2}{12}$$

Notes

- The uniform extends to the continuous case the idea of equally likely outcomes.
- If $Y \sim U[0, 1]$, then $a + (b - a)Y \sim U[a, b]$

Exponential Distribution Denoted $X \sim Exp(\lambda)$:

- sample space: $x \geqslant 0$
- pdf:

$$f(x) = \begin{cases} \lambda e^{-\lambda y} & \text{for } y \geqslant 0 \\ 0 & otherwise \end{cases}$$

- cdf:

$$F(x) = \int_{-\infty}^{x} f(y)dy = \begin{cases} 1 - e^{-\lambda x} & \text{for } x \geqslant 0 \\ 0 & \text{for } x < 0 \end{cases}$$

- moments:

$$\begin{aligned} E(X) &= 1/\lambda \\ Var(X) &= 1/\lambda^2 \\ M_X(t) &= \lambda/(\lambda - t), \quad t < \lambda \end{aligned}$$

Lecture 26: Oct 25

Last time

- Common Continuous Distributions

Today

- Common Continuous Distributions

Exponential Distribution Denoted $X \sim Exp(\lambda)$:

- sample space: $x \geqslant 0$
- pdf:
$$f(x) = \begin{cases} \lambda e^{-\lambda y} & \text{for } y \geqslant 0 \\ 0 & otherwise \end{cases}$$
- cdf:
$$F(x) = \int_{-\infty}^{x} f(y)dy = \begin{cases} 1 - e^{-\lambda x} & \text{for } x \geqslant 0 \\ 0 & \text{for } x < 0 \end{cases}$$
- moments:
$$\begin{aligned} E(X) &= 1/\lambda \\ Var(X) &= 1/\lambda^2 \\ M_X(t) &= \lambda/(\lambda - t), \quad t < \lambda \end{aligned}$$

Interpretation The exponential can be derived as the waiting time between Poisson events. Suppose that the number of events in a unit interval of time follows a Poisson(λ) distribution. Then, let Y be the time until the first event.

$$\Pr(Y > t) = \Pr(0 \text{ events in } [0, t])$$

and the number of events in $[0, t]$ follows a Poisson distribution with parameter λt. Therefore,

$$\Pr(Y > t) = e^{-\lambda t}.$$

The cdf of Y is

$$F(t) = 1 - \Pr(Y > t) = 1 - e^{-\lambda t}$$

and hence the density is $f(t) = \lambda e^{-\lambda t}$.

Alternative parameterization Many books write the density as

$$f(y) = \begin{cases} \frac{1}{\theta} e^{-y/\theta} & \text{for } y \geqslant 0 \\ 0 & otherwise \end{cases}$$

so that $E(Y) = \theta$ and $Var(Y) = \theta^2$. In this case $\theta = 1/\lambda$ is called the *mean parameter*, while $\lambda = 1/\theta$ is called the *rate parameter*.

Memoryless property The exponential has a memoryless property, just like the geometric.

$$\Pr(Y > s + t | Y > t) = \Pr(Y > s)$$

Same interpretation as the geometric for continuous time:

- The probability of an event in a time interval depends only on the length of the interval, not the absolute time of the interval.
- The underlying Poisson process is stationary: the rate λ is constant. (In the geometric case, the probability, p of getting an event in every discrete time unit is constant).

Shifted exponential Let $X \sim Exp(\lambda)$ and $Y = X + v, v \in \mathbb{R}$. Then, Y has the *shifted exponential distribution* with pdf:

$$f(y) = \begin{cases} \lambda e^{-(y-v)\lambda} & \text{for } y \geqslant v \\ 0 & otherwise \end{cases}$$

Interpretation:

- $v > 0$: Event is delayed
- $v < 0$: The news of the event is delayed

Does the shifted expoenential maintain the memoryless property?

Double exponential The *double exponential distribution* is formed by reflecting an exponential distribution around zero. It has pdf:

$$f(x) = \frac{1}{2}\lambda e^{-\lambda|x|}, \quad x \in \mathbb{R}$$

Laplace distribution Suppose X has the above distribution with $\lambda = 1$. Now let $Y = \sigma X + \mu, \mu \in \mathbb{R}$ (shifting) and $\sigma > 0$ (scaling). Then Y has the *Laplace distribution* with pdf:

$$f_Y(y) = \frac{1}{2\sigma} \exp\left(-\frac{|y-\mu|}{\sigma}\right)$$

with moments

$$EY = \mu, \quad Var(Y) = 2\sigma^2$$

The Laplace distribution provides an alternative to the normal for centered data with fatter tails but all finite moments.

Normal Distribution Introduced by De Moivre (1667 - 1754) in 1733 as an approximation to the binomial. Later studied by Laplace and others as part of the Central Limit Theorem. Gauss derived the normal as a suitable distribution for outcomes that could be thought of as sums of many small deviations.

- Sample space: $\mathbb{R} = (-\infty, \infty)$

- pdf: For $Y \sim N(\mu, \sigma^2)$,

$$f(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-\mu)^2}{2\sigma^2}} \quad -\infty < y < \infty$$

- cdf: There is no closed form.
- When $\mu = 0$ and $\sigma = 1$, the distribution is called *standard normal*:

$$\Phi(y) = \Pr(Y \leqslant y), \quad \Phi(-y) = 1 - \Phi(y)$$

- Mean:

$$EY = \mu$$

- Variance:

$$Var(Y) = E(Y - \mu)^2 = \sigma^2$$

- Higher central moments:

$$E(Y-\mu)^m = \begin{cases} \frac{m!}{2^{m/2}(m/2)!}\sigma^m & m \text{ is even} \\ 0 & m \text{ is odd} \end{cases}$$

- In particular:

$$\mu_3 = E(Y-\mu)^3 = 0 \text{(Skewness)}$$
$$\mu_4 = E(Y-\mu)^4 = 3\sigma^4$$

- Moment generating function:

$$M_Y(t) = \exp(\mu t + \sigma^2 t^2/2)$$

Standardization

$$Y \sim N(\mu, \sigma^2) \iff Z = \frac{Y-\mu}{\sigma} \sim N(0,1)$$

Shifting and scaling:

$$Z \sim N(0,1) \iff Y = \sigma Z + \mu \sim N(\mu, \sigma^2)$$

Notes

- Normal distribution is useful in many practical settings. E.g. measurement error.
- Plays an important role in *sampling distributions* in *large samples*, since the Central Limit Theorem syas that the sums of independent identically distributed random variables are approximately normal
- There are many important distributions that can be derived from functions of normal random variables (e.g. χ^2, t, F). We will briefly present the pdf's and sample spaces of these distributions.

Lecture 27: Oct 28

Last time

- Common Continuous Distribution

Today

- Common Continuous Distribution

Standardization

$$Y \sim N(\mu, \sigma^2) \iff Z = \frac{Y - \mu}{\sigma} \sim N(0, 1)$$

Shifting and scaling:

$$Z \sim N(0, 1) \iff Y = \sigma Z + \mu \sim N(\mu, \sigma^2)$$

Notes

- Normal distribution is useful in many practical settings. E.g. measurement error.
- Plays an important role in *sampling distributions* in *large samples*, since the Central Limit Theorem syas that the sums of independent identically distributed random variables are approximately normal
- There are many important distributions that can be derived from functions of normal random variables (e.g. χ^2, t, F). We will briefly present the pdf's and sample spaces of these distributions.

χ^2 distribution If $Z \sim N(0, 1)$, then $X = Z^2$ has the χ^2 distribution with 1 degree of freedom. More generally, we have the χ^2 distribution with v degrees of freedom with pdf:

$$f(x) = \frac{(x/2)^{\frac{v}{2}-1} e^{-x/2}}{2\Gamma(v/2)}, \quad x > 0$$

where $\Gamma(a)$ is the complete gamma function,

$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} dx$$

The $\chi^2(v)$ distribution is a special case of the gamma distribution, so it is easier to derive its properties from the gamma.

Facts about the Gamma function

- $\Gamma(a + 1) = a\Gamma(a), a > 0$
- $\Gamma(1) = 1$
- $\Gamma(n) = (n - 1)!$
- $\Gamma(1/2) = \sqrt{\pi}$

Lecture 28: Oct 30

Last time

- Common Continuous Distribution

Today

- Common Continuous Distribution

Student's t and F distributions Y has a t_v distribution (t with v degrees of freedom) if its pdf can be written as:

$$f(y) = \frac{\Gamma\left[(v+1)/2\right]}{\sqrt{v\pi}\Gamma(v/2)} \frac{1}{(1+y^2/v)^{(v+1)/2}}, \quad -\infty < y < \infty$$

Y has an $F(v_1, v_2)$ distribution if its pdf can be written as:

$$f(y) = \frac{(v_1/v_2)\Gamma\left[(v_1+v_2)/2\right](v_1 y/v_2)^{v_1/2-1}}{\Gamma(v_1/2)\Gamma(v_2/2)(1+v_1 y/v_2)^{(v_1+v_2)/2}}, \quad 0 \leqslant y < \infty$$

There are many important properties and relationships between these three distributions (e.g., χ^2_k is the distribution of the sum of the squares of k independent standard normals).

Gamma distribution Notation: $Y \sim Gamma(a, \lambda)$.

- pdf:

$$f(y) = \frac{\lambda e^{-\lambda y}(\lambda y)^{a-1}}{\Gamma(a)}, \quad y \geqslant 0$$

  where $\Gamma(a)$ is the gamma function,

$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} dx$$

- cdf: In general, there is no closed form, unless a is an integer.
- moments:

$$E(Y) = a/\lambda$$
$$Var(Y) = a/\lambda^2$$

- MGF:

$$M_Y(t) = \left(\frac{1}{1-t/\lambda}\right)^a, \quad t < \theta$$

Another parameterization Same as the exponential distribution, we can let $\beta = \frac{1}{\lambda}$, then we have

- pdf:
$$f(y) = \frac{y^{a-1}e^{-y/\beta}}{\Gamma(a)\beta^a}, \quad y \geqslant 0$$
- moments:
$$EX = \alpha\beta$$
$$Var(X) = \alpha\beta^2$$
- MGF:
$$M_Y(t) = \left(\frac{1}{1-t\beta}\right)^a, \quad t < \frac{1}{\beta}$$

Notes:

- The special case $a = 1$ corresponds to an $exponential(\lambda)$
- The parameter a is known as the *shape parameter*, since it most influences the peakedness of the distribution.
- The parameter β is called the *scale parameter* since most of its influence is on the spread of the distribution.
- The special case $Gamma(a = n/2, \lambda = 1/2)$, for integer n, corresponds to the χ^2_n distribution with n degrees of freedom.
- The gamma distribution can be derived as the sum of a independent $exponential(\lambda)$ distributions.

Lecture 29: Nov 1

Last time

- Common Continuous Distribution

Today

- Common Continuous Distribution

Beta distribution Notation: $Y \sim Beta(a, b)$.

- Sample space: $[0, 1]$
- pdf:
$$f(y) = \frac{y^{a-1}(1-y)^{b-1}}{B(a,b)}, \quad 0 \leqslant y \leqslant 1$$
where $B(a, b)$ is the Beta function,
$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},$$
and $\Gamma(a)$ is the gamma function. Note that if a and b are integers, then $B(a, b)$ can be calculated in closed form.
- cdf: In general, there is no closed form, except if a and b are integers.
- moments:
$$EY = \frac{a}{a+b}$$
$$Var(Y) = \frac{ab}{(a+b)^2(a+b+1)}$$
The beta distribution is very flexible, and can take a wide variety of shapes by varying its parameters.
- Special case: $Beta(1, 1) = U(0, 1)$.

Omitted distributions: Weibull distribution, and Cauchy distribution.

Lecture 30: Nov 4

Last time

- Common Continuous Distribution

Today

- Common Continuous Distribution
- Distribution Families

Location and Scale families

Let Z be a continuous random variable with pdf $f(z)$. Define the class of rvs

$$X_{\mu,\sigma} = \sigma Z + \mu, \quad \mu \in \mathbb{R}, \sigma > 0$$

Then

1. $X_{\mu,\sigma}$ has pdf
$$f_{\mu,\sigma}(x) = \frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right)$$
2. $$E(X) = \sigma E(Z) + \mu, \quad Var(X) = \sigma^2 Var(Z)$$
3. The variable $Z = X_{0,1}$ is called the *generator* and is a member of the class.

Location families and scale families

- The family of pdfs $f_{\mu,\sigma}(x)$ is called a *location-scale* family where μ is called the *location parameter*, and σ is called the *scale parameter*.
- The family of pdfs
$$f_{\mu,1}(x) = f(x-\mu)$$
with $\sigma = 1$ is called a *location* family.
- The family of pdfs
$$f_{0,\sigma}(x) = \frac{1}{\sigma} f\left(\frac{x}{\sigma}\right)$$
with $\mu = 0$ is called a *scale* family.

Example (Exponential location family) Let $f(x) = e^{-x}, x \geqslant 0$, and $f(x) = 0, x < 0$. To form a location family we replace x with $x - \mu$ to obtain

$$f(x|\mu) = \begin{cases} e^{-(x-\mu)} & x - \mu \geqslant 0 \\ 0 & x - \mu < 0 \end{cases}$$
$$= \begin{cases} e^{-(x-\mu)} & x \geqslant \mu \\ 0 & x < \mu \end{cases}$$

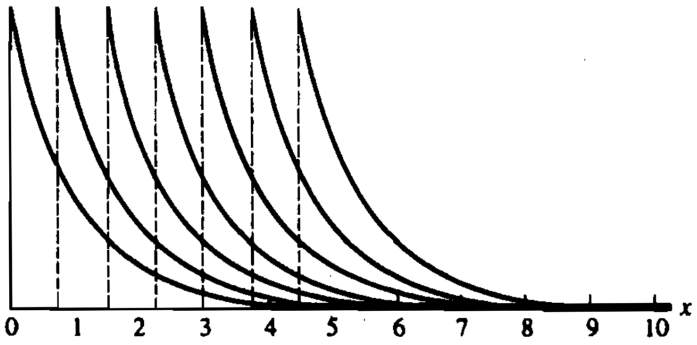

Figure 3.5.2. ***Exponential location densities***

Figure 29.3: Figure 3.5.2. Exponential location densities.

As shown in the above graph, the densities are shifted. Now the positive part of the density starts at μ rather than at 0. If X measures time, then μ might be restricted to be nonnegative so that X will be positive with probability 1 for every value of μ. In this type of model, where μ denotes a bound on the range of X, μ is sometimes called a *threshold parameter*.

The effect of introducing the scale parameter σ is either to stretch ($\sigma > 1$) or to contract ($\sigma < 1$) the graph of $f(x)$ while still maintaining the same basic shape of the graph. This is illustrated in the Figure below.

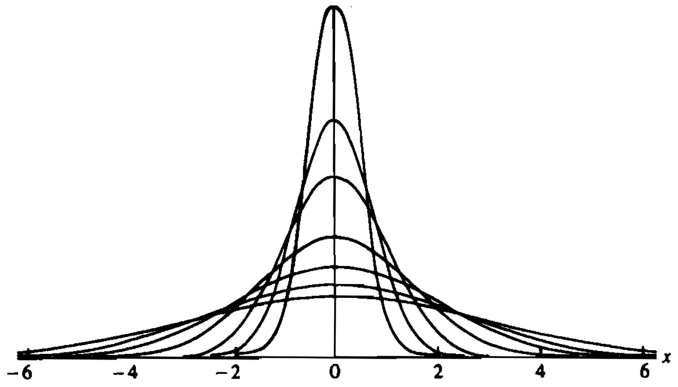

Figure 29.4: Figure 3.5.3. Members of the same scale family

Exponential Families A family of pdfs or pmfs with vector parameter $\boldsymbol{\theta}$ is called an *exponential family* if it can be expressed as

$$f(x|\boldsymbol{\theta}) = h(x)c(\boldsymbol{\theta})exp\left(\sum_{j=1}^{k} w_j(\boldsymbol{\theta})t_j(x)\right), \quad x \in S \subset \mathbb{R} \tag{1}$$

where S is not defined in terms of $\boldsymbol{\theta}$, $h(x)$, $c(\boldsymbol{\theta}) \geqslant 0$ and the functions are just functions of the parameters specified; i.e. h is free of $\boldsymbol{\theta}$, $c(\boldsymbol{\theta})$ is free of x, etc...

Examples:

- One-dimensional: Exponential, Poisson
- Two-dimensional: Gaussian

Exponential family parameterizations are unique except for multiplying constant factors.

Example: Gaussian Let $f(x|\mu, \sigma^2)$ be the $n(\mu, \sigma^2)$ family of pdfs, where $\boldsymbol{\theta} = (\mu, \sigma)$. Then

$$\begin{aligned} f(x|\mu, \sigma^2) &= \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \\ &= \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\mu^2}{2\sigma^2}\right) \exp\left(-\frac{x^2}{2\sigma^2} + \frac{\mu x}{\sigma^2}\right) \end{aligned}$$

Thus

$$\begin{aligned} h(x) = \tfrac{1}{\sqrt{2\pi}} \quad & c(\mu, \sigma) = \tfrac{1}{\sigma} \exp\left(-\tfrac{\mu^2}{2\sigma^2}\right) \\ w_1(\mu, \sigma) = -\tfrac{1}{2\sigma^2} \quad & w_2(\mu, \sigma) = \tfrac{\mu}{\sigma^2} \\ t_1(x) = x^2 \quad & t_2(x) = x \end{aligned}$$

The parameter space is $(\mu, \sigma^2) \in \mathbb{R} \times (0, \infty)$.

Lecture 31: Nov 6

Last time

- Common Continuous Distribution
- Location Scale Families

Today

- Exponential Family

Exponential Families A family of pdfs or pmfs with vector parameter $\boldsymbol{\theta}$ is called an *exponential family* if it can be expressed as

$$f(x|\boldsymbol{\theta}) = h(x)c(\boldsymbol{\theta})exp\left(\sum_{j=1}^{k} w_j(\boldsymbol{\theta})t_j(x)\right), \quad x \in S \subset \mathbb{R} \tag{2}$$

where S is not defined in terms of $\boldsymbol{\theta}$, $h(x)$, $c(\boldsymbol{\theta}) \geqslant 0$ and the functions are just functions of the parameters specified; i.e. h is free of $\boldsymbol{\theta}$, $c(\boldsymbol{\theta})$ is free of x, etc...

Examples:

- One-dimensional: Exponential, Poisson
- Two-dimensional: Gaussian

Exponential family parameterizations are unique except for multiplying constant factors.

Example: Gaussian Let $f(x|\mu, \sigma^2)$ be the $n(\mu, \sigma^2)$ family of pdfs, where $\boldsymbol{\theta} = (\mu, \sigma)$. Then

$$\begin{aligned} f(x|\mu,\sigma^2) &= \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \\ &= \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{\mu^2}{2\sigma^2}\right)\exp\left(-\frac{x^2}{2\sigma^2}+\frac{\mu x}{\sigma^2}\right) \end{aligned}$$

Thus

$$\begin{aligned} h(x) = \tfrac{1}{\sqrt{2\pi}} \quad & c(\mu,\sigma) = \tfrac{1}{\sigma}\exp\left(-\tfrac{\mu^2}{2\sigma^2}\right) \\ w_1(\mu,\sigma) = -\tfrac{1}{2\sigma^2} \quad & w_2(\mu,\sigma) = \tfrac{\mu}{\sigma^2} \\ t_1(x) = x^2 \quad & t_2(x) = x \end{aligned}$$

The parameter space is $(\mu, \sigma^2) \in \mathbb{R} \times (0, \infty)$.

Example: Binomial Let $f(x|p)$ be the *binomial*(n, p), $0 < p < 1$ family of pmfs.

$$\begin{aligned} f(x|p) &= \binom{n}{x}p^x(1-p)^{n-x} = \binom{n}{x}(1-p)^n\left[\frac{p}{1-p}\right]^x \\ &= \binom{n}{x}(1-p)^n\exp\left[\log\left(\frac{p}{1-p}\right)x\right] \end{aligned}$$

Thus,

$$h(x) = \binom{n}{x}, \quad x = 0, \ldots, n \quad w_1(p) = \log\left(\tfrac{p}{1-p}\right)$$
$$c(p) = (1-p)^n, 0 < p < 1 \quad t_1(x) = x$$

Note that this works when p is considered the parameter, while n is fixed. Also, p cannot be 0 or 1. Otherwise, the range changes.

More examples The following distributions belong to Exponential families:

- Continuous: exponential, Gaussian, gamma, beta, χ^2
- Discrete: Poisson, geometric, binomial (fixed # trials), negative binomial (fixed # successes)

The following distributions do not belong to exponential families:

- Continuous: t, F, unifrom
  For example, $X \sim U(0, \theta)$, $f_X(x) = \theta^{-1} I_{0<x<\theta}$
- Discrete: uniform, hypergeometric

Theorem If X is a random variable with pdf or pmf of the form 2, then

$$E\left(\sum_{i=1}^{k} \frac{\partial w_i(\boldsymbol{\theta})}{\partial \theta_j} t_i(X)\right) = -\frac{\partial}{\partial \theta_j} \log c(\boldsymbol{\theta})$$
$$Var\left(\sum_{i=1}^{k} \frac{\partial w_i(\boldsymbol{\theta})}{\partial \theta_j} t_i(X)\right) = -\frac{\partial^2}{\partial \theta_j^2} \log c(\boldsymbol{\theta}) - E\left(\sum_{i=1}^{k} \frac{\partial^2 w_i(\boldsymbol{\theta})}{\partial \theta_j^2} t_i(X)\right).$$

Although these equations may look formidable, when applied to specific cases they can work out quite nicely. Their advantage is that we can replace integration or summation by differentiation, which is often more straightforward.

Example (Normal exponential family) Let $f(x|\mu, \sigma^2)$ be the $n(\mu, \sigma^2)$ family of pdfs, where $\boldsymbol{\theta} = (\mu, \sigma), -\infty < \mu < \infty, \sigma > 0$. Then

$$\begin{aligned} f(x|\mu, \sigma^2) &= \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \\ &= \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\mu^2}{2\sigma^2}\right) \exp\left(-\frac{x^2}{2\sigma^2} + \frac{\mu x}{\sigma^2}\right) \end{aligned}$$

Define

$$\theta_1 = \tfrac{1}{\sigma^2} > 0, \quad \theta_2 = \tfrac{\mu}{\sigma^2} \in \mathbb{R}$$

Then

$$f_X(x) = \frac{\sqrt{\theta_1}}{\sqrt{2\pi}} \exp\left(-\frac{\theta_2^2}{2\theta_1}\right) \exp\left(-\theta_1 \frac{x^2}{2} + \theta_2 x\right)$$

and

$$
\begin{aligned}
&h(x) = 1 \text{ for all } x; \\
&c(\boldsymbol{\theta}) = c(\theta_1, \theta_2) = \exp\left(-\tfrac{\theta_2^2}{2\theta_1}\right), \quad (\theta_1, \theta_2) \in (0, \infty) \times \mathbb{R} \\
&w_1(\boldsymbol{\theta}) = \theta_1 \qquad\qquad\qquad\qquad t_1(x) = -x^2/2 \\
&w_2(\boldsymbol{\theta}) = \theta_2 \qquad\qquad\qquad\qquad t_2(x) = x
\end{aligned}
$$

Therefore, by the above theorem

$$
\begin{aligned}
E(X) &= -\frac{\partial}{\partial \theta_2} \log c(\boldsymbol{\theta}) = \frac{\theta_2}{\theta_1} = \mu \\
Var(X) &= -\frac{\partial^2}{\partial \theta_2^2} \log c(\boldsymbol{\theta}) = -\frac{1}{\theta_1} = \sigma^2
\end{aligned}
\tag{3}
$$

Lecture 32: Nov 15

Last time

- Midterm 2

Today

- Multiple Random Variables (Chapter 4)

Joint and Marginal Distributions

In previous lectures, we have discussed probability models and computation of probability for events involving only one random variable. These are called *univariate models*.

In an experimental situation, it would be very unusual to observe only the value of one random variable. For example, in an experiment designed to gain information about some health characteristics of a population of people, the body weights of several people in the population might be measured. These different weights would be observations on difference random variables, one for each person measured. Multiple observations could also arise because several physical characteristics were measured on each person. Thus, we need to know how to describe and use probability models that deal with more than one random variable at a time.

Definition: An n-dimensional random vector $\boldsymbol{X} = (X_1, \ldots, X_n)$ is a function from a sample space S into $\mathbb{R}^n$.

- Each coordinate X_i is a random variable.
- The random vector is associated with a probability space $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), F)$.
- For each Borel set B,
$$\Pr\{\boldsymbol{X} \in B\} = \Pr\{\boldsymbol{X}^{-1}(B)\} \tag{4}$$
where
$$\boldsymbol{X}^{-1}(B) = \{w : \boldsymbol{X}(w) \in B\}$$

Example (Bivariate random variable) A fair coin is flipped 3 times. Define the random vector (X, Y) where X represents the number of heads on the last toss and Y the total number of heads. Then, the probabilities of various outcomes are given in the following table:

Outcome	(x, y)	Pr(*outcome*)
(H, H, H)	(1, 3)	1/8
(H, T, H), (T, H, H)	(1, 2)	2/8
(H, H, T)	(0, 2)	1/8
(T, T, H)	(1, 1)	1/8
(T, H, T), (H, T, T)	(0, 1)	2/8
(T, T, T)	(0, 0)	1/8

Definition Two random variables X and Y are said to be jointly *discrete* if there is an associated *joint probability mass function*,

$$f_{X,Y}(x, y) = \Pr\{X = x, Y = y\}$$

which sums to 1 over a finite or possibly countable combinations of x and y for which $f_{X,Y}(x, y) > 0$, i.e.,

$$\sum_{x,y} f_{X,Y}(x, y) = 1$$

From this, one can also obtain the marginal pmfs of X and Y as follows:

$$f_X(x) = \Pr(X = x) = \sum_y f_{X,Y}(x, y)$$
$$f_Y(y) = \Pr(Y = y) = \sum_x f_{X,Y}(x, y)$$

Lecture 33: Nov 18

Last time

- Midterm exam 2

Today

- Multiple Random Variables (Chapter 4)

Example (Bivariate random variable) A fair coin is flipped 3 times. Define the random vector (X, Y) where X represents the number of heads on the last toss and Y the total number of heads. Then, the probabilities of various outcomes are given in the following table:

Outcome	(x, y)	$\Pr(outcome)$
(H, H, H)	(1, 3)	1/8
(H, T, H), (T, H, H)	(1, 2)	2/8
(H, H, T)	(0, 2)	1/8
(T, T, H)	(1, 1)	1/8
(T, H, T), (H, T, T)	(0, 1)	2/8
(T, T, T)	(0, 0)	1/8

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$$f_X(x) = \Pr(X = x) = \sum_y f_{X,Y}(x, y)$$
$$f_Y(y) = \Pr(Y = y) = \sum_x f_{X,Y}(x, y)$$

Example Back to the fair coin example again. From the definition, we can construct the joint pmf of X and Y:

		Y			
		0	1	2	3
X	0	1/8	1/4	1/8	0
	1	0	1/8	1/4	1/8

The marginal distributions of X and Y are also easy to find. Note: Marginals do not determine joint pmf.

Bivariate cdfs Whether they are discrete or continuous or some combination of the two, we can always define the *joint cdf*. For $n = 2$, the *bivariate cumulative distribution function* is

$$F_{X,Y}(x, y) = \Pr\{X \leqslant x, Y \leqslant y\}$$

Properties:

- $F_{X,Y}(x, y) \geqslant 0$
- $F_{X,Y}(\infty, \infty) = 1$
- $F_{X,Y}(-\infty, y) = F_{X,Y}(x, -\infty) = 0$
- $F_{X,Y}(-\infty, -\infty) = 0$
- F is non-decreasing and right-continuous in each variable separately.

Joint probabilities All joint probability statements about X and Y can be answered in terms of their joint cdf:

$$\begin{aligned}\Pr(x_1 < X \leqslant x_2, y_1 < Y \leqslant y_2) = \\ F_{X,Y}(x_2, y_2) + F_{X,Y}(x_1, y_1) - F_{X,Y}(x_1, y_2) - F_{X,Y}(x_2, y_1)\end{aligned}$$

Example

$$\Pr(X > x, Y > y) = 1 - F_X(x) - F_Y(y) + F_{X,Y}(x, y)$$

Note: To ensure that a bivariate function $F(x, y)$ is a proper cdf, it must satisfy all the properties mentioned above and the rectangular property above.

Marginal distributions From $F_{X,Y}$, we can derive the univariate distribution functions for X and Y. These are generally called *marginal distributions*.

$$\begin{aligned}F_X(x) = \Pr\{X \leqslant x\} = \Pr\{X \leqslant x, Y \leqslant \infty\} = F_{X,Y}(x, \infty) \\ F_Y(y) = \Pr\{Y \leqslant y\} = \Pr\{X < \infty, Y \leqslant y\} = F_{X,Y}(\infty, y)\end{aligned}$$

Note: Although we can obtain $F_X(x)$ and $F_Y(y)$ from the joint cdf, we cannot do the reverse.

Continuous Bivariate RVs The random variables X and Y are said to be *jointly continuous* if there exists a function $f_{X,Y}(x, y)$, such that for any Borel set B of 2-tuples in $\mathbb{R}^2$,

$$\Pr\{(X, Y) \in B\} = \int\int_{(x,y)\in B} f_{X,Y}(x, y)dxdy.$$

The function $f_{X,Y}(x, y)$ is called the *joint probability density function* for X and Y. It follows in this case that

$$\begin{aligned}F_{X,Y}(x, y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(s, t)dtds, \\ f_{X,Y}(x, y) = \frac{\partial^2 F(x, y)}{\partial x \partial y}\end{aligned}$$

Lecture 34: Nov 20

Last time

- Multiple Random Variables (Chapter 4)

Today

- Course Evaluations
- Conditional Distributions
- Independence

Continuous Bivariate RVs The random variables X and Y are said to be *jointly continuous* if there exists a function $f_{X,Y}(x,y)$, such that for any Borel set B of 2-tuples in $\mathbb{R}^2$,

$$\Pr\{(X,Y) \in B\} = \int\int_{(x,y)\in B} f_{X,Y}(x,y)dxdy.$$

The function $f_{X,Y}(x,y)$ is called the *joint probability density function* for X and Y. It follows in this case that

$$F_{X,Y}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(s,t)dtds,$$
$$f_{X,Y}(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y}$$

Properties of the bivariate pdf

- $f_{X,Y}(x,y) \geqslant 0$
- $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy = 1$
- $f_{X,Y}(x,y)$ is not a probability, but can be thought of as a relative probability of (X,Y) falling into a small rectangle located at (x,y):

$$\Pr\{x < X \leqslant x+dx, y < Y \leqslant y+dy\} \approx f(x,y)dxdy$$

- The *marginal probability density functions* for X and Y can be obtained as

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy$$
$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx$$

Example 1

$$F_{X,Y}(x,y) = xy \qquad 0 < x \leqslant 1, 0 < y \leqslant 1$$
$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y} =$$
$$f_X(x) =$$
$$f_Y(y) =$$

Example 2

$$F_{X,Y}(x,y) = x - x\log\frac{x}{y} \qquad 0 < x \leqslant y \leqslant 1$$
$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y} =$$
$$f_X(x) =$$
$$f_Y(y) =$$

Note: Once we have $f_X(x)$ and $f_Y(y)$, we can obtain $F_X(x)$ and $F_Y(y)$ directly. Double check: $F_X(x) = F_{X,Y}(x, \infty)$.

Conditional Distributions

Conditional Distributions - Discrete Recall if A and B are two events, the probability of A conditional on B is:

$$\Pr(A|B) = \frac{\Pr(A,B)}{\Pr(B)}$$

Defining the events $A = \{Y = y\}$ and $B = \{X = x\}$, it follows that

$$\begin{aligned}\Pr\{Y = y|X = x\} &= \frac{\Pr(X = x, Y = y)}{\Pr(X = x)} \\ &= \frac{f_{X,Y}(x,y)}{f_X(x)} \\ &= f_{Y|X}(y|x)\end{aligned}$$

This is called the *conditional probability mass function* of Y given X.

Example: Discrete Back to the fair coin example. From the joint pmf of X and Y, we can derive all the conditional pmfs:

		Y			
		0	1	2	3
X	0	1/8	1/4	1/8	0
	1	0	1/8	1/4	1/8

Lecture 35: Nov 22

Last time

- Multiple Random Variables (Chapter 4)

Today

- Course Evaluations (7/33)
- Independence

Conditional Distribution - Continuous If $F(x, y)$ is absolutely continuous, we define the conditional density of X given Y as:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x, y)}{f_Y(y)}, \text{ if } f_Y(y) > 0$$

Example 1

$$\begin{array}{lll}
F_{XY}(x, y) = xy & 0 < x < 1, \quad 0 < y < 1 & \\
f_{XY}(x, y) = 1 & 0 < x < 1, \quad 0 < y < 1 & \\
f_X(x) = 1 & 0 < x < 1 & \\
f_Y(y) = 1 & 0 < y < 1 & \\
f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = 1 & 0 < x < 1 & (0 < y < 1) \\
f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = 1 & 0 < y < 1 & (0 < x < 1)
\end{array}$$

Note: Here we get that the conditional densities are the same as the marginals. This means X and Y are independent.

Example 2

$$\begin{array}{lll}
F_{XY}(x, y) = x - x \log \frac{x}{y} & 0 < x \leqslant y \leqslant 1 & \\
f_{XY}(x, y) = 1/y & 0 < x \leqslant y \leqslant 1 & \\
f_X(x) = -\log x & 0 < x \leqslant 1 & \\
f_Y(y) = 1 & 0 < y \leqslant 1 & \\
f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = 1/y & 0 < x \leqslant y & (0 < y \leqslant 1) \\
f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = -\frac{1}{y \log x} & x \leqslant y \leqslant 1 & (0 < x \leqslant 1)
\end{array}$$

- Y is marginally uniform, but not conditionally uniform.
- X is conditionally uniform, but not marginally uniform.

Independent Random Variables

Independence The random variable X and Y are said to be *independent* if for any two Borel sets A and B,

$$\Pr(X \in A, Y \in B) = \Pr(X \in A) \Pr(Y \in B)$$

All events defined in terms of X are independent of all events defined in terms of Y.

Using the Kolmogorov axioms of probability, it can be shown that X and Y are independent if and only if $\forall(x, y)$ (except possibly for sets of probability 0)

$$F_{X,Y}(x, y) = F_X(x)F_Y(y)$$

or in terms of pmfs (discrete) and pdfs (continuous)

$$f_{X,Y}(x, y) = f_X(x)f_Y(y)$$

Checking independence

- A necessary condition for independence of X and Y is that their joint pdf/pmf has positive probability on a rectangular domain.
- If the domain is rectangular, one can try to write the joint pdf/pmf as a product of functions of x and y only.

Example Two points are selected randomly on a line of length a so as to be on opposite sides of the mid-point of the line. Find the probability that the distance between them is less than $a/3$.

Solution:
Let X be the coordinate of a point selected randomly in $[0, a/2]$ and Y be the coordinate of a point selected randomly in $[a/2, a]$. Assume X and Y are independent and uniform over its interval. The joint density is

$$f_{X,Y}(x, y) = 4/a^2, \quad 0 \leqslant x \leqslant a/2, a/2 \leqslant y \leqslant a$$

Therefore, the solution is

$$\Pr(Y - X < a/3) =$$

Lecture 36: Dec 2

Last time

- Multiple Random Variables (Chapter 4)

Today

- Course Evaluations (8/33)
- Expectation
- Final exam format
  - Final exam will be take home
  - Open book, open note, not open internet
  - Final exam will be released on Friday (12/06/2024) right after class
  - Final exam due 23:59 pm on Friday 12/13/2024.
  - Scan and submit your exam via email with a single pdf file.
  - Send your email to both your instructor and your TA.
  - Submitted exams should be human-readable to receive non-zero scores.

Example: Buffon's Needle A table is ruled with lines distance 1 unit apart. A needle of length $L \leqslant 1$ is thrown randomly on the table. What is the probability that the needle intersects a line?
Solution:
Define two random variables:

- X: distance from low end of the needle to the nearest line above
- θ: angle from the vertical to the needle.

By "random", we assume X and θ are independent, and

$$X \sim U(0,1) \quad \text{and} \quad \theta \sim U[-\pi/2, \pi/2].$$

This means that

$$f_{X,\theta}(x,\theta) = 1/\pi, \quad 0 \leqslant x \leqslant 1, -\pi/2 \leqslant \theta \leqslant \pi/2$$

For the needle to intersect a line, we need $X < L\cos(\theta)$.

Expectations of Independent RVs (Theorem 4.2.10) Let X and Y be independent rvs.

- For any $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$,

$$\Pr(X \in A, Y \in B) = \Pr(X \in A)\Pr(Y \in B)$$

  i.e., the events $\{X \in A\}$ and $\{Y \in B\}$ are independent.

- Let $g(x)$ be a function only of x and $h(y)$ be a function only of y. Then

$$E\left[g(X)h(Y)\right] = \left[Eg(X)\right]\left[Eh(Y)\right]$$

Proof:

$$\begin{aligned}
E\left[g(X)h(Y)\right] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)f_{XY}(x,y)dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)f_X(x)f_Y(y)dxdy \\
&= \left(\int_{-\infty}^{\infty} g(x)f_X(x)dx\right)\left(\int_{-\infty}^{\infty} h(y)f_Y(y)dy\right) \\
&= \left[Eg(X)\right]\left[Eh(Y)\right]
\end{aligned}$$

Example X, Y are independent

$$\begin{aligned}
E(X^2Y^3) &= (EX^2)(EY^3) \\
E(Y^2Y^3) &\neq (EY^2)(EY^3)
\end{aligned}$$

Lecture 37: Dec 4

Last time

- Expectation

Today

- Course Evaluations (9/33)
- Transformations

Bivariate Transformation

Functions of random variables Let (X, Y) be a bivariate rv with known distributions. Define (U, V) by

$$U = g_1(X, Y), \quad V = g_2(X, Y)$$

Probability mapping For any Borel set $B \subset \mathbb{R}^2$,

$$\Pr\left[(U, V) \in B\right] = \Pr\left[(X, Y) \in A\right]$$

where A is the inverse mapping of B, such that

$$A = \{(x, y) \in \mathbb{R}^2 : (g_1(x, y), g_2(x, y)) \in B\}.$$

The inverse is well defined even if the mapping is not bijective.

Example Let $g_1(x, y) = x, g_2(x, y) = x^2 + y^2$.

Discrete RVs Suppose that (X, Y) is a discrete rv, i.e., the pmf is positive on a countable set $\mathcal{A}$. Then (U, V) is also discrete and takes values on a countable set $\mathcal{B}$. Define

$$A_{u,v} = \{(x, y) \in \mathcal{A} : g_1(x, y) = u, g_2(x, y) = v\}$$

Then

$$f_{UV}(u, v) = \Pr(U = u, V = v) = \sum_{(x,y) \in A_{u,v}} f_{XY}(x, y)$$

Sum of two independent Poissons Let $X \sim Poisson(\lambda_1)$, $Y \sim Poisson(\lambda_2)$, independent, and define

$$U = X + Y, \quad V = Y$$

- (X, Y) takes values in $\mathcal{A} = \{0, 1, 2, \dots\} \times \{0, 1, 2, \dots\}$
- (U, V) takes values on $\mathcal{B} = \{(u, v) : v = 0, 1, 2, \dots, u = v, v + 1, v + 2, \dots\}$.
- For a particular (u, v), $A_{uv} = \{(x, y) \in \mathcal{A} : x + y = u, y = v\} = (u - v, u)$.

The joint pmf of U and V is

$$f_{UV}(u,v) = f_{XY}(u-v,v) = \frac{e^{-\lambda_1}\lambda_1^{u-v}}{(u-v)!}\frac{e^{-\lambda_2}\lambda_2^{v}}{(v)!}$$

The distribution of $U = X + Y$ is the marginal

$$\begin{aligned} f_U(u) &= \sum_{v=0}^{u} \frac{e^{-\lambda_1}\lambda_1^{u-v}}{(u-v)!}\frac{e^{-\lambda_2}\lambda_2^{v}}{(v)!} \\ &= \frac{e^{-(\lambda_1+\lambda_2)}}{u!}\sum_{v=0}^{u}\binom{u}{v}\lambda_1^{u-v}\lambda_2^{v} \\ &= \frac{e^{-(\lambda_1+\lambda_2)}}{u!}(\lambda_1+\lambda_2)^u \end{aligned}$$

We obtain that U is Poisson with parameter $\lambda = \lambda_1 + \lambda_2$.

Bivariate Transformations of Continuous RVs Suppose (X, Y) is continuous and the joint transformation

$$u = g_1(x,y), \quad v = g_2(x,y)$$

is one-to-one and differentiable. Define the inverse mapping

$$x = h_1(u,v), \quad y = h_2(u,v)$$

Then

$$f_{UV}(u,v) = f_{XY}\left(h_1(u,v), h_2(u,v)\right)||J(u,v)||$$

where $J(u,v)$ is the Jacobian of the transformation $(x,y) \to (u,v)$ given by

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

Example: Rotation of a bivariate normal vector Let $X \sim N(0,1)$, $Y \sim N(0,1)$, independent. Define the rotation

$$\begin{aligned} U &= X\cos\theta - Y\sin\theta \\ V &= X\sin\theta + Y\cos\theta \end{aligned}$$

for fixed θ. Then $U \sim N(0,1)$, $V \sim N(0,1)$, independent.

Proof:
The range of (X,Y) is $\mathbb{R}^2$. The range of (U,V) is $\mathbb{R}^2$. Need the inverse transformation

$$\begin{aligned} X &= U\cos\theta + V\sin\theta \\ Y &= -U\sin\theta + V\cos\theta \end{aligned}$$

with Jacobian

$$J(u,v) = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

and its determinant is

$$|J(u,v)| = 1.$$

The joint pdf of (X,Y) is

$$f_{XY}(x,y) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2} \cdot \frac{1}{\sqrt{2\pi}}e^{-y^2/2} = \frac{1}{2\pi}e^{-(x^2+y^2)/2}$$

The joint pdf of (U,V) is

$$\begin{aligned} f_{UV}(u,v) &= \frac{1}{2\pi}e^{-\left[(u\cos\theta+v\sin\theta)^2+(-u\sin\theta+v\cos\theta)^2\right]/2} \cdot |1| \\ &= \frac{1}{2\pi}e^{-(u^2+v^2)/2} = \frac{1}{\sqrt{2\pi}}e^{-u^2/2} \cdot \frac{1}{\sqrt{2\pi}}e^{-v^2/2} \end{aligned}$$

so $U \sim N(0,1)$, $V \sim N(0,1)$, and U and V are independent.

Functions of independent random variables (Theorem 4.3.5) Let X and Y be independent rvs. Let $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ be functions. Then the random variables $U = g(X)$ and $V = h(Y)$ are independent.

Sum of two independent rvs Suppose X and Y are independent. What is the distribution of $Z = X + Y$? In general:

$$F_Z(z) = \Pr(X + Y \leqslant z) = \Pr(\{(x,y) \text{ such that } x + y \leqslant z\})$$

Various approaches:

- bivariate transformation method (continuous and discrete)
- Discrete convolution

$$f_Z(z) = \sum_{x+y=z} f_X(x)f_Y(y) = \sum_x f_X(x)f_Y(z-x)$$

- Continuous convolution (Section 5.2)
- MGF method (continuous and discrete)

Example (Sum of two independent Poissons) Define X, Y to be two independent random variables having Poisson distributions with parameters λ_i, $i = 1, 2$. Then:

$$f_{X,Y}(x,y) = \frac{e^{-\lambda_1}\lambda_1^x}{x!}\frac{e^{-\lambda_2}\lambda_2^y}{y!}, x, y = 0, 1, 2, \ldots$$

The distribution of $S = X + Y$ is

$$\begin{aligned} f_S(s) &= \sum_{x=0}^{s} \frac{e^{-\lambda_1}\lambda_1^x}{x!}\frac{e^{-\lambda_2}\lambda_2^{s-x}}{(s-x)!} \\ &= \frac{e^{-(\lambda_1+\lambda_2)}}{s!}\sum_{x=0}^{s}\binom{s}{x}\lambda_1^x\lambda_2^{s-x} \\ &= \frac{e^{-(\lambda_1+\lambda_2)}}{s!}(\lambda_1+\lambda_2)^s \end{aligned}$$

Again, S is Poisson with parameter $\lambda = \lambda_1 + \lambda_2$.

Moment generating function (Theorem 4.2.12) Let X and Y be independent rvs with mgfs $M_X(\cdot)$ and $M_Y(\cdot)$, respectively. Then the mgf of $Z = X + Y$ is

$$M_Z(t) = M_X(t)M_Y(t)$$

Proof:

$$\begin{aligned} M_Z(t) &= E\exp(Zt) && = E\{\exp[(X+Y)t]\} \\ &= E[\exp(Xt)\exp(Yt)] && = E[\exp(Xt)]\cdot E[\exp(Yt)] \\ &= M_X(t)M_Y(t) \end{aligned}$$

Corollary: If X and Y are independent and $Z = X - Y$,

$$M_Z(t) = M_X(t)M_Y(-t)$$

Example (sum of two independent Poissons) Suppose $X \sim Poisson(\lambda_X)$ and $Y \sim Poisson(\lambda_Y)$ and put $Z = X + Y$. Then, $Z \sim Poisson(\lambda_X + \lambda_Y)$. *Proof:*

$$\begin{aligned} M_Z(t) &= \exp\left[\lambda_X(e^t-1)\right]\exp\left[\lambda_Y(e^t-1)\right] \\ &= \exp\left[(\lambda_X+\lambda_Y)(e^t-1)\right] \end{aligned}$$

Example (sum of two independent normals) Suppose $X \sim N(\mu_x, \sigma_x^2)$ and $Y \sim N(\mu_y, \sigma_y^2)$ and X and Y are independent and $Z = X + Y$. Then

$$Z \sim N(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2)$$

Proof:

$$\begin{aligned} M_Z(t) &= \exp\left(\mu_x t + \frac{1}{2}\sigma_x^2 t^2\right)\exp\left(\mu_y t + \frac{1}{2}\sigma_y^2 t^2\right) \\ &= \exp\left[(\mu_x+\mu_y)t + \frac{1}{2}(\sigma_x^2+\sigma_y^2)t^2\right] \end{aligned}$$

Example (sum of two independent gammas) Suppose $X \sim \Gamma(\alpha_x, \beta)$ and independently $Y \sim \Gamma(\alpha_y, \beta)$. Let $Z = X + Y$. Then $Z \sim \Gamma((\alpha_x + \alpha_y), \beta)$.
Proof:

$$\begin{aligned} M_Z(t) &= \left(\frac{1}{1-\beta t}\right)^{\alpha_x} \left(\frac{1}{1-\beta t}\right)^{\alpha_y} \\ &= \left(\frac{1}{1-\beta t}\right)^{\alpha_x+\alpha_y} \end{aligned}$$

Remember that

- If $\alpha = 1$ we have an exponential with parameter β.
- If $\alpha = n/2$ and $\beta = 2$, we have a $\chi^2(n)$ (with n d.f.). The above result states that $\chi^2(n_1) + \chi^2(n_2) = \chi^2(n_1 + n_2)$.

Covariance and Correlation Let X and Y be two random variables with respective means μ_X, μ_Y and variances $\sigma_X^2 > 0$ and $\sigma_Y^2 > 0$, all assumed to exist.

- The *covariance* of X and Y is

$$Cov(X, Y) = E\left[(X - \mu_X)(Y - \mu_Y)\right] = \sigma_{XY}$$

- The *correlation* between X and Y is

$$Cor(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)Var(Y)}}$$

also written as

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = E\left[\left(\frac{X - \mu_X}{\sigma_X}\right)\left(\frac{Y - \mu_Y}{\sigma_Y}\right)\right]$$

Properties Let c be a constant:

1. $Cov(X, X) = Var(X)$, $Cor(X, X) = 1$
2. $Cov(X, Y) = Cov(Y, X)$, $Cor(X, Y) = Cor(Y, X)$
3. $Cov(X, c) = 0$, $Cor(X, c) = 0$
4. $Cov(X, Y) = E(XY) - E(X)E(Y)$

5. Let $X_c = X - \mu_X$, $Y_c = Y - \mu_Y$. Then

$$\begin{aligned} Cov(X, Y) &= Cov(X_c, Y_c) = E(X_c Y_c) \\ Cor(X, Y) &= Cor(X_c, Y_c) \end{aligned}$$

6. Let $\tilde{X} = (X - \mu_X)/\sigma_X$, $\tilde{Y} = (Y - \mu_Y)/\sigma_Y$. Then,

$$Cor(X, Y) = Cor(\tilde{X}, \tilde{Y}) = Cov(\tilde{X}, \tilde{Y}) = E(\tilde{X}\tilde{Y})$$

Independent vs. Uncorrelated

- X and Y are called *uncorrelated* iff

$$Cov(X,Y) = 0 \quad \text{or equivalently} \quad \rho_{XY} = 0$$

- If X and Y are independent and $Cov(X,Y)$ exists, then $Cov(X,Y) = 0$.
- If X and Y are uncorrelated, this does **not** imply that they are independent.

Example $X \sim U[-1,1]$, $Y = X^2$. Then $Cov(X,Y) = 0$ but X, Y are not independent.

Correlation coefficient For any random variables X and Y,

1. $-1 \leqslant \rho_{XY} \leqslant 1$
2. $|\rho_{XY}| = 1$ if and only if $\exists a \neq 0$ and b such that

$$\Pr(Y = aX + b) = 1.$$

if $\rho_{XY} = 1$ then $a > 0$, and if $\rho_{XY} = -1$, then $a < 0$.

proof:
Let $\tilde{X} = (X - \mu_X)/\sigma_X$, $\tilde{Y} = (Y - \mu_Y)/\sigma_Y$. Then $Cor(X,Y) = E(\tilde{X}\tilde{Y})$,

1.
$$\begin{aligned} 0 \leqslant E(\tilde{X} - \tilde{Y})^2 = 1 + 1 - 2E(\tilde{X}\tilde{Y}) \quad &\Rightarrow \quad E(\tilde{X}\tilde{Y}) \leqslant 1 \\ 0 \leqslant E(\tilde{X} + \tilde{Y})^2 = 1 + 1 + 2E(\tilde{X}\tilde{Y}) \quad &\Rightarrow \quad -1 \leqslant E(\tilde{X}\tilde{Y}) \end{aligned}$$

2.
$$\begin{aligned} \rho_{XY} = 1 \iff \Pr(\tilde{Y} = \tilde{X}) = 1 \quad &\Rightarrow \quad a > 0 \\ \rho_{XY} = -1 \iff \Pr(\tilde{Y} = -\tilde{X}) = 1 \quad &\Rightarrow \quad a < 0 \end{aligned}$$

Random Samples

Definition The random variables $X_1, \ldots, X_n$ are called a *random sample of size n from the population* $f(x)$ if $X_1, \ldots, X_n$ are mutually independent and identically distributed (iid) random variables with the same pdf or pmf $f(x)$.

If $X_1, \ldots, X_n$ are iid, then their joint pdf or pmf is

$$f(x_1, \ldots, x_n) = f(x_1)f(x_2)\ldots f(x_n) = \prod_{j=1}^{n} f(x_j)$$

Statistics Let $X_1, \ldots, X_n$ be a random sample and let $T(x_1, \ldots, x_n)$ be a function defined on $\mathbb{R}^n$. Then the random variable $Y = T(X_1, \ldots, X_n)$ is called a *statistic*. The probability distribution of Y is called the *sampling distribution* of Y.

Note: T is only a function of $(x_1, \ldots, x_n)$, no parameters.

Examples

$$\begin{aligned} \text{sample mean} \quad & \bar{X} = \tfrac{1}{n}\sum_{j=1}^{n} X_j \\ \text{sample variance} \quad & S^2 = \tfrac{1}{n-1}\sum_{j=1}^{n}(X_j - \bar{X})^2 \\ \text{sample standard deviation} \quad & S = \sqrt{S^2} \\ \text{minimum} \quad & X_{(1)} = \min_{1 \leqslant i \leqslant n} X_i \end{aligned}$$

Properties Let $x_1, \ldots, x_n$ be n numbers and define

$$\bar{x} = \frac{1}{n}\sum_{j=1}^{n} x_j, \quad s^2 = \frac{1}{n-1}\sum_{j=1}^{n}(x_j - \bar{x})^2$$

Then

$$\begin{aligned} \min_a \sum_{j=1}^{n}(x_j - a)^2 &= \sum_{j=1}^{n}(x_j - \bar{x})^2 \\ (n-1)s^2 &= \sum_{j=1}^{n}(x_j - \bar{x})^2 = \sum_{j=1}^{n} x_j^2 - n\bar{x}^2 \end{aligned}$$

Residuals Lemma: Let $X_1, \ldots, X_n$ be a random sample from a population with mean μ and variance σ^2. Define the residuals $R_i = X_i - \bar{X}$. Then

$$\begin{aligned} E(R_i) = 0, \quad & Var(R_i) = \tfrac{n-1}{n}\sigma^2 \\ Cov(R_i, \bar{X}) = 0, \quad & Cov(R_i, R_j) = -\sigma^2/n \text{ if } i \neq j \end{aligned}$$

Theorem Let $X_1, \ldots, X_n$ be a random sample from a population with mgf $M_X(t)$. Then the mgf of the sample mean is

$$M_{\bar{X}}(t) = [M_X(t/n)]^n$$

Convergence

Convergence in Probability A sequence of random variables $X_1, \ldots, X_n$ *converges in probability* to a random variable X, denoted

$$X_n \xrightarrow{p} X$$

if for every $\epsilon > 0$,

$$\lim_{n\to\infty} \Pr(|X_n - X| < \epsilon) = 1$$

or equivalently

$$\lim_{n\to\infty} \Pr(|X_n - X| > \epsilon) = 0$$

In other words, X_n is more and more likely to be close to X, or less and less likely to be far from X.

Example Let $X_n = X + \epsilon_n$, where $\epsilon_n \sim N(0, 1/n)$ and X is an arbitrary random variable. Then, as $n \to \infty$,

$$X_n \xrightarrow{p} X$$

Weak law of large numbers (WLLN) Let $Y_1, \ldots, Y_n$ be iid with common mean μ and variance σ^2. Then, as $n \to \infty$,

$$\bar{Y}_n = \frac{1}{n} \sum_{j=1} Y_j \xrightarrow{p} \mu$$

Proof:
The proof is quite simple, being a straightforward application of Chebychev's Inequality. We have, for every $\epsilon > 0$,

$$\Pr(|\bar{Y}_n - \mu| \geqslant \epsilon) = \Pr(|\bar{Y}_n - \mu|^2 \geqslant \epsilon^2) \leqslant \frac{E(\bar{Y} - \mu)^2}{\epsilon^2} = \frac{Var(\bar{Y})}{\epsilon^2} = \frac{\sigma^2}{n\epsilon^2} \to 0 \text{ as } n \to \infty$$

Convergence in Distribution A sequence of random variables $X_1, \ldots, X_n$ *converges in distribution* to a random variable X, denoted

$$X_N \xrightarrow{d} X$$

if

$$\lim_{n \to \infty} F_{X_n}(x) = F_X(x)$$

This is also called *convergence in law* or *weak convergence*. In other words, the distribution of X_n is closer and closer to the distribution of X.

Relation between "in distribution" and "in probability" Theorem:

1. Convergence in probability implies convergence in distribution:

$$X_n \xrightarrow{p} X \Rightarrow X_n \xrightarrow{d} X$$

2. Suppose $X_n \xrightarrow{d} X$ where X has a degenerate distribution, i.e. $\Pr\{X = a\} = 1$ for some $a \in \mathbb{R}$. Then,

$$X_n \xrightarrow{d} a \Rightarrow X_n \xrightarrow{p} a$$

Convergence in Distribution via Convergence of Mgfs Theorem: Suppose the mgf $M_n(t)$ of Y_n exists for $|t| < h$, and the mgf $M(t)$ of Y exists for $|t| < h_1 < h$. Then,

$$Y_n \xrightarrow{d} Y \iff \lim_{n \to \infty} M_n(t) = M(t), \quad |t| < h_1$$

Example Let $X_\lambda \sim Poisson(\lambda)$. Then, as $\lambda \to \infty$,

$$\frac{X_\lambda - \lambda}{\lambda} \xrightarrow{p} 0$$
$$\frac{X_\lambda - \lambda}{\sqrt{\lambda}} \xrightarrow{d} N(0, 1)$$

Central Limit Theorem Let $X_1, X_2, \ldots, X_n$ be a sequence of iid random variables whose mgfs exist in a neighborhood of 0 (that is, $M_{X_i}(t)$ exists for $|t| < h$, for some positive $h > 0$). Let $EX_i = \mu$ and $Var(X_i) = \sigma^2 > 0$. (Both μ and σ^2 are finite since the mgf exists) Define $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Let $G_n(x)$ denote the cdf of $\sqrt{n}(\bar{X}_n - \mu)/\sigma$. Then, for any x, $-\infty < x < \infty$,

$$\lim_{n\to\infty} G_n(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy;$$

that is, $\sqrt{n}(\bar{X}_n - \mu)/\sigma$ has a limiting standard normal distribution, in other words, $\sqrt{n}(\bar{X}_n - \mu)/\sigma \xrightarrow{d} N(0,1)$

Proof:
Define $Y_i = (X_i - \mu)/\sigma$, and let $M_Y(t)$ denote the common mgf of Y_is, which exists for $|t| < \sigma h$ and $M_Y(t) = M_{\frac{1}{\sigma}X_i - \mu/\sigma}(t) = e^{-\frac{\mu}{\sigma}t} M_X(\frac{t}{\sigma})$. Since

$$\frac{\sqrt{n}(\bar{X}_n)}{\sigma} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Y_i,$$

we have,

$$\begin{aligned} M_{\sqrt{n}(\bar{X}_n - \mu)/\sigma}(t) &= M_{\sum_{i=1}^{n} Y_i/\sqrt{n}}(t) \\ &= M_{\sum_{i=1}^{n} Y_i}(t/\sqrt{n}) \\ &= \left[M_Y(t/\sqrt{n})\right]^n. \end{aligned}$$

We now expand $M_Y(t/\sqrt{n})$ in a Taylor series (power series) around 0.

$$M_Y(\frac{t}{\sqrt{n}}) = \sum_{k=0}^{\infty} M_Y^{(k)}(0) \frac{(t/\sqrt{n})^k}{k!},$$

where $M_Y^{(k)}(0) = (d^k/dt^k) M_Y(t)|_{t=0}$. Since the mgfs exist for $|t| < h$, the power series expansion is valid if $t < \sqrt{n}\sigma h$.

Using the facts that $M_Y^{(0)} = 1$, $M_Y^{(1)} = 0$, and $M_Y^{(2)} = 1$ (by construction, the mean and variance of Y are 0 and 1), we have

$$M_Y(\frac{t}{\sqrt{n}}) = 1 + \frac{(t/\sqrt{n})^2}{2!} + R_Y(\frac{t}{\sqrt{n}}),$$

where R_Y is the remainder term in the Taylor expansion such that

$$\lim_{n\to\infty} \frac{R_Y(t/\sqrt{n})}{(t/\sqrt{n})^2} = 0.$$

Therefore, for any fixed t, we can write

$$\begin{aligned} \lim_{n\to\infty} \left[M_Y(\frac{t}{\sqrt{n}})\right]^n &= \lim_{n\to\infty} \left[1 + \frac{(t/\sqrt{n})^2}{2!} + R_Y(\frac{t}{\sqrt{n}})\right]^n \\ &= \lim_{n\to\infty} \left[1 + \frac{1}{n}\left(\frac{t^2}{2} + nR_Y(\frac{t}{\sqrt{n}})\right)\right]^n \\ &= e^{t^2/2} \end{aligned}$$