Math 3070/6070 Homework 2
Due: Sept 15th, 2025

1. (1.24) Two players, A and B, alternately and independently flip a coin and the first player to obtain a head wins. Assume player A flips first.
   (a) If the coin is fair, what is the probability that A wins?
   (b) Suppose that $\Pr(\text{head}) = p$, not necessarily $\frac{1}{2}$. What is the probability that A wins?
   (c) Show that for all p, $0 < p < 1$, $\Pr(\text{A wins}) > \frac{1}{2}$. (*Hint:* Try to write $\Pr(\text{A wins})$ in ters of the events $E_1, E_2, \ldots$, where $E_i = \{\text{head first appears on } i\text{th toss}\}$.)
2. (1.34) Two litters of a particular rodent species have been born, one with two brown-haired and one gray-haired (litter 1), and the other with three brown-haired and two gray-haired (litter 2). We select a litter at random and then select an offspring at random from the selected litter.
   (a) What is the probability that the animal chosen is brown-haired?
   (b) Given that a brown-haired offspring was selected, what is the probability that the sampling was from litter 1?
3. (1.36) If the probability of hitting a target is $\frac{1}{5}$, and ten shots are fired independently, what is the probability of the target being hit at least twice? What is the conditional probaiblity that the target is hit at least twice, given that it is hit at least once?
4. (1.38) Prove each of the following statements. (Assuming that any conditioning event has positive probability.)
   (a) If $\Pr(B) = 1$, then $\Pr(A|B) = \Pr(A)$ for any A.
   (b) If $A \subset B$, then $\Pr(B|A) = 1$ and $\Pr(A|B) = \Pr(A)/\Pr(B)$.
   (c) If A and B are mutually exclusive, then
   $$\Pr(A|A \cup B) = \frac{\Pr(A)}{\Pr(A) + \Pr(B)}.$$
   (d) $\Pr(A \cap B \cap C) = \Pr(A|B \cap C)\Pr(B|C)\Pr(C)$
5. (1.44) Standardized tests provide an interesting application of probability theory. Suppose first that a test consists of 20 multiple-choice questions, each with 4 possible answers. If the student guesses on each question, then the taking of the exam can be modeled as a sequence of 20 independent events. Find the probability that the student gets at least 10 questions correct, given that he is guessing.
6. **Credit: Dr. Jacqueline M. Hughes-Oliver at North Carolina State University**
   Tests for impurity must be conducted on k different samples of a substance. The old method was to test each sample individually, which requires k tests in all. A new method is now proposed as follows:
   1. Combine all k samples into a single sample and test the sample for impurity.

2. If the group test result is negative (showing no impurities), then we can claim all k samples are pure, and only 1 test was needed.
3. If the group test result is positive (showing impurities), then all k samples must be tested individually.

Suppose that each sample has the same probability, p, of being pure (and hence the test would be negative), and that the k samples are independent.

(a) How many tests are possible under the new system in order to classify all samples? That is, list the possible numbers of tests.

(b) Find the probability that only 1 test is needed to classify all samples.

(c) Find the probability that exactly $k + 1$ tests are needed to classify all samples.